

Induced surfaces and their integrable dynamics.

II. Generalized Weierstrass representations in 4D spaces and deformations via DS hierarchy.

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Abstract

Extensions of the generalized Weierstrass representation to generic surfaces in 4D Euclidean and pseudo-Euclidean spaces are given. Geometric characteristics of surfaces are calculated. It is shown that integrable deformations of such induced surfaces are generated by the Davey-Stewartson hierarchy. Geometrically these deformations are characterized by the invariance of an infinite set of functionals over surface. The Willmore functional (the total squared mean curvature) is the simplest of them. Various particular classes of surfaces and their integrable deformations are considered.

1 Introduction

Surfaces and their deformations (dynamics) were for a long time the subjects of intensive study both in mathematics and physics. Theory of immersions and deformations of surfaces has been a significant part of the classical differential geometry (see *e.g.* [1-3]). Various methods to describe immersions and deformations have been developed. This subject continues to be an important part of the contemporary differential geometry (see *e.g.* [4-6]).

In physics, the dynamics of interfaces, surfaces, fronts is a key ingredient in a number of interesting phenomena from hydrodynamics, propagation of flame fronts, growth of crystals, deformations of membranes to world-sheets and their dynamics in the string theory (see *e.g.* [7-9]). Such dynamics could be modelled by nonlinear partial differential equations. Analytic methods to study surfaces, their properties and deformations are of great interest both in mathematics and physics (see [7-9] and recent papers [10-11]).

A general method to construct surfaces via the solutions of linear differential equations and their deformations via the corresponding nonlinear integrable equations has been proposed in [12-13]. The two basic examples considered in [13] were given by: 1) the generalized Weierstrass formulae for generic surfaces conformally immersed into \mathbb{R}^3 and deformations via the modified Veselov-Novikov equation; 2) the Lelievre formula for surfaces in \mathbb{R}^3 referred to asymptotic lines and integrable deformations via the Nizhnik-Veselov-Novikov equation.

The generalized Weierstrass representation proposed in [12-13] has been proved to be an effective tool to study generic surfaces in \mathbb{R}^3 and their deformations. In differential geometry its use has allowed to obtain several interesting results both of local and global character, in particular, for the Willmore functional $W = \int H^2 [dS]$ where H is the mean curvature of the surface (see *e.g.* [17-22]). In physics, it has been applied to study of various problems in the theory of liquid membranes, 2D gravity and string theory [17, 23-27]. In the

string theory the functional $W = \int H^2 [dS]$ is known as the Polyakov extrinsic action and in membrane theory it is the Helfrich free energy [7-9]. Deformations constructed via the Lelievre formula and the Nizhnik-Veselov equation have occurred to be an interesting class of deformations of surfaces in affine geometry [28].

An extension of the Weierstrass representation to multidimensional spaces would be of a great interest. In physics, a strong motivation lies in the Polyakov string integral over surfaces in multidimensional spaces [7-9]. Theory of immersion of surfaces into four-dimensional spaces is an important part of the contemporary differential geometry too [4-6, 29-32].

In this paper we present extensions of the generalized Weierstrass representation to the cases of generic surfaces conformally immersed into the four-dimensional spaces \mathbb{R}^4 , $\mathbb{R}^{3,1}$ and $\mathbb{R}^{2,2}$ with the metrics $g_{ik} = \text{diag}(1, 1, 1, 1)$, $g_{ik} = \text{diag}(1, 1, 1, -1)$ and $g_{ik} = \text{diag}(1, 1, -1, -1)$, respectively. A basic linear system consists of a couple of the two-dimensional Dirac equations while the formulae for immersions, the induced metric, mean curvature and Willmore functional are of the type similar to that of the \mathbb{R}^3 case.

Integrable deformations of surfaces are generated by the Davey-Stewartson (DS) hierarchy of 2+1-dimensional soliton equations. These deformations of surfaces inherit all remarkable properties of the soliton equations. Geometrically, such deformations are characterized by the invariance of an infinite set of functionals over surfaces. The simplest of them is given by the Willmore functional.

We consider both space-like and time-like surfaces in the case of pseudo-Euclidean spaces. Various particular classes of surfaces, including minimal and superminimal surfaces, immersions with constant mean curvature and their deformations are discussed. It is shown that one special class of the 1+1-dimensional reductions gives rise to an integrable motions of curves on the three-dimensional sphere \mathbb{S}^3 and hyperboloids described by the nonlinear Schroedinger

equation.

Extensions of the generalized Weierstrass formulae to the four-dimensional Riemann spaces are discussed too.

The present paper can be considered as the second part of the paper [13]. In fact, the DS inducing of surfaces (in \mathbb{R}^3) and their deformations via the DS hierarchy have been mentioned in [13] (section 10, pp. 41-42) as one of possibilities. Here we elaborate this case in detail.

Note that some results of this paper has been presented briefly in [33]. The Weierstrass type representations for particular classes of surfaces have been discussed in [30, 34] and recently in [35].

The paper is organized as follows. In section 2 we remind for convenience some results concerning the generalized Weierstrass formulae for surfaces in \mathbb{R}^3 . In section 3 we present the Weierstrass representation for generic surfaces conformally immersed into \mathbb{R}^4 . Space-like surfaces in $\mathbb{R}^{2,2}$, $\mathbb{R}^{3,1}$ are considered in section 4 while in section 5 we concentrate on time-like surfaces in pseudo-Euclidean spaces. Surfaces in four-dimensional Riemann spaces are discussed in section 6. Integrable deformations of surfaces in 4D spaces via the DS hierarchy are described in section 7. Explicit expressions for deformations of surfaces are given in section 8. Particular classes of surfaces and their deformations are considered in section 9. The one-dimensional reduction of the Weierstrass representation and corresponding surfaces are discussed in section 10. Appendix contains some basic facts about the DS hierarchy.

2 Generalized Weierstrass formulae for surfaces in \mathbb{R}^3 .

A generalization of the Weierstrass formulae to generic surfaces in \mathbb{R}^3 proposed by one of the authors in 1993 (see [12] and [13]) starts with the linear system

(two-dimensional Dirac equation)

$$\begin{aligned}\psi_z &= p\varphi \quad , \\ \varphi_{\bar{z}} &= -p\psi\end{aligned}\tag{2.1}$$

where ψ and φ are complex-valued functions of $z, \bar{z} \in \mathbb{C}$ and $p(z, \bar{z})$ is a real-valued function. Then one defines the three real-valued functions $X^1(z, \bar{z})$, $X^2(z, \bar{z})$ and $X^3(z, \bar{z})$ by the formulae

$$\begin{aligned}X^1 + iX^2 &= i \int_{\Gamma} (\bar{\psi}^2 dz' - \bar{\varphi}^2 d\bar{z}') \quad , \\ X^1 - iX^2 &= i \int_{\Gamma} (\varphi^2 dz' - \psi^2 d\bar{z}') \quad , \\ X^3 &= - \int_{\Gamma} (\bar{\psi}\varphi dz' + \psi\bar{\varphi} d\bar{z}')\end{aligned}\tag{2.2}$$

where Γ is an arbitrary contour in \mathbb{C} . In virtue of (2.1) the *r.h.s.* in (2.2) do not depend on the choice of Γ . If one now treats $X^i(z, \bar{z})$ as the coordinates in \mathbb{R}^3 then the formulae (2.1), (2.2) define a conformal immersion of surface into \mathbb{R}^3 with the induced metric of the form

$$ds^2 = u^2 dz d\bar{z} = (|\psi|^2 + |\varphi|^2)^2 dz d\bar{z} \quad ,\tag{2.3}$$

with the Gauss curvature

$$K = -\frac{4}{u^2} [\log u]_{z\bar{z}}\tag{2.4}$$

and the mean curvature

$$H = 2\frac{p}{u} \quad .\tag{2.5}$$

At $p = 0$ one gets minimal surfaces and the formulae (2.2) are reduced to the old Weierstrass formulae.

Another analog of the Weierstrass formulae for surfaces of prescribed (non zero) mean curvature have been proposed earlier by Kenmotsu in [36]. The Kenmotsu representation is given by

$$\vec{X} = \text{Re} \left[\int_{\Gamma} \eta \vec{\phi} dz' \right]\tag{2.6}$$

where $\vec{\phi} = [1 - f^2, i(1 + f^2), 2f]$ and the functions f and η obey the following compatibility condition

$$(\log \eta)_{\bar{z}} = -\frac{2\bar{f}f_{\bar{z}}}{1 + |f|^2} \quad . \quad (2.7)$$

Here and below the bar denotes the complex conjugation. Then the mean curvature H is

$$H = -\frac{2f_{\bar{z}}}{\bar{\eta}(1 + |f|^2)^2} \quad . \quad (2.8)$$

It was proved in [9] that any surface in \mathbb{R}^3 can be presented in such a form. This representation of surfaces deals basically with the Gauss map for generic surface in \mathbb{R}^3 [31].

It turned out that the Kenmotsu formulae (2.6), (2.7) and the generalized Weierstrass formulae (2.1), (2.2) are equivalent to each other. The relation between the functions (f, η) and (ψ, φ) is the following [14]

$$f = i\frac{\bar{\psi}}{\varphi} \quad , \quad \eta = i\varphi^2 \quad (2.9)$$

and

$$p = -\frac{\eta f_{\bar{z}}}{\sqrt{\eta\bar{\eta}}(1 + |f|^2)} \quad . \quad (2.10)$$

So all results proved for the Kenmotsu formulae [36] and associated Gauss map [31] in \mathbb{R}^3 are valid also for the generalized Weierstrass formulae (2.2). In particular, it implies immediately that any surface \mathbb{R}^3 in can be represented via (2.1)-(2.2).

Though the representations (2.1), (2.2) and (2.6), (2.7) are equivalent, the former provides us certain advantages. They are mainly due to the fact that in the generalized Weierstrass formulae the functions ψ and φ obey the linear equations (2.1) while for the Kenmotsu formulae the nonlinear constraint (2.7) is difficult to deal with. This circumstance has allowed to simplify essentially an analysis that had lead to several interesting results both of local and global character [15-21]. It occurred, in particular, that the Willmore functional (see

e.g. [4]) or the Helfrich-Polyakov action (see [7-9]) $W = \int \vec{H}^2 [dS]$ has a very simple form: $W = 4 \int p^2 dx dy$ ($z = x + iy$) [14-15].

One of the advantages of the generalized Weierstrass formulae (2.1), (2.2) is that they allow to construct a new class of deformations of surfaces via the modified Veselov-Novikov equation [12-13]. The characteristic feature of these integrable deformations is that the Willmore functional remains invariant [14-15]. Thus, the generalized Weierstrass representation (2.2) has been proved to be an effective tool to study surfaces in \mathbb{R}^3 and their deformations.

We would like to emphasize that the idea to generate surfaces via solutions of linear equations is, in fact, the old idea of the classical differential geometry as it was already noted in [13]. In [3] one can find the two representations of these type in addition to the Weierstrass formulae. The first is given by the Lelievre formula

$$\begin{aligned}\vec{X}_\xi &= \vec{\nu} \times \vec{\nu}_\xi \quad , \\ \vec{X}_\eta &= -\vec{\nu} \times \vec{\nu}_\eta\end{aligned}\tag{2.11}$$

where $\vec{\nu} = (\nu_1, \nu_2, \nu_3)$ are three linearly independent solutions of the equation

$$\vec{\nu}_{\xi\eta} + p\vec{\nu} = 0\tag{2.12}$$

and the $\nu_i(\xi, \eta)$'s and $p(\xi, \eta)$ are scalar functions. The Lelievre formula defines immersion of a surface into \mathbb{R}^3 ($\vec{X} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$) parametrized by asymptotic lines $\xi = \text{const}$ and $\eta = \text{const}$. The Lelievre formula is well-known in affine geometry of surfaces.

Another example is provided by the equation

$$\theta_{\eta\xi} - (\log \lambda)_\eta \theta_\xi - \lambda^2 \theta = 0\tag{2.13}$$

where ξ and η are real variables and λ is a real-valued function. It is stated in [3] that two solutions of (2.13) define, via certain integral formulae, a surface in \mathbb{R}^3 parametrized by minimal lines, but no calculation of the metric and curvature

is given. This example, seems, was forgotten completely until it has been found during the preparation of the second paper [13] on the generalized Weierstrass formulae. The representation (2.13) is rather close to that of (2.1)-(2.2). Indeed, equation (2.13) can be rewritten as the system

$$\begin{aligned}\theta_\xi &= \lambda\varphi \quad , \\ \varphi_\eta &= \lambda\theta\end{aligned}\tag{2.14}$$

where φ is a new function. If one takes two solutions (θ, φ) and $(\tilde{\theta}, \tilde{\varphi})$ of the system (2.14) then the formulae given in [3] (pp. 82) take the form

$$\begin{aligned}X^1 + iX^2 &= \int (\theta^2 d\eta + \varphi^2 d\xi) \quad , \\ X^1 - iX^2 &= \int (\tilde{\theta}^2 d\eta + \tilde{\varphi}^2 d\xi) \quad , \\ X^3 &= i \int (\theta\tilde{\theta} d\eta + \varphi\tilde{\varphi} d\xi) \quad .\end{aligned}\tag{2.15}$$

However, in contrast to the representation (2.1), (2.2), the formulae (2.14), (2.15) do not define a real surface in \mathbb{R}^3 .

We would like to note that some results in [37] and [38] were close to the generalized Weierstrass representation (2.2). In [37] a formula similar to (2.2) for constant mean curvature surfaces has been discussed. In [38] the system (2.1) had appeared within the quaternionic description of surfaces in \mathbb{R}^3 (formula (2.19) of [38]). However in [38] it was accompanied by another two equations (equation (2.16) of [38]) which are indispensable in the Sym's type approach. So the meaning of the system (2.1), seems, has been missed. The generalized Weierstrass type formulae admit also a beautiful formulation within the spinor representation of surfaces [39-40].

3 The Weierstrass representation for surfaces in

\mathbb{R}^4

An extension of the representation (2.1), (2.2) to the four-dimensional Euclidean space is quite natural. Let ψ_1, φ_1 and ψ_2, φ_2 be solutions of the systems

$$\begin{aligned} \psi_{1z} &= p\varphi_1 \quad , & \psi_{2z} &= \bar{p}\varphi_2 \quad , \\ \varphi_{1\bar{z}} &= -\bar{p}\psi_1 & \varphi_{2\bar{z}} &= -p\psi_2 \quad . \end{aligned} \quad (3.1)$$

Equations (3.1) imply that

$$(\psi_1\psi_2)_z = -(\varphi_1\varphi_2)_{\bar{z}} \quad , \quad (\psi_1\bar{\varphi}_2)_z = (\varphi_1\bar{\psi}_2)_{\bar{z}} \quad . \quad (3.2)$$

As a consequence there are four functions $X^i(z, \bar{z})$ ($i = 1, 2, 3, 4$) such that

$$\begin{aligned} dX^1 &= \frac{1}{2} (\bar{\psi}_1\bar{\psi}_2 - \varphi_1\varphi_2) dz + c.c. \\ dX^2 &= \frac{i}{2} (\bar{\psi}_1\bar{\psi}_2 + \varphi_1\varphi_2) dz + c.c. \\ dX^3 &= \frac{1}{2} (\varphi_1\bar{\psi}_2 + \psi_1\bar{\varphi}_2) dz + c.c. \\ dX^4 &= \frac{i}{2} (\bar{\psi}_1\varphi_2 - \varphi_1\bar{\psi}_2) dz + c.c. \end{aligned} \quad (3.3)$$

where *c.c.* denotes a complex conjugation of the previous term. We treat now these functions $X^i(z, \bar{z})$ as the coordinates of surfaces in \mathbb{R}^4 . For components of induced metric

$$g_{zz} = \sum_{i=1}^4 (X_z^i)^2 = \overline{g_{\bar{z}\bar{z}}} \quad , \quad g_{z\bar{z}} = \sum_{i=1}^4 (X_z^i X_{\bar{z}}^i) \quad (3.4)$$

one gets

$$g_{zz} = g_{\bar{z}\bar{z}} = 0 \quad (3.5)$$

and

$$g_{z\bar{z}} = \frac{1}{2} \left(|\psi_1|^2 + |\varphi_1|^2 \right) \left(|\psi_2|^2 + |\varphi_2|^2 \right) \quad . \quad (3.6)$$

Further, two normal vectors \vec{N}_1, \vec{N}_2 are

$$\vec{N}_1 = \sqrt{\frac{|\varphi_1|^2 |\varphi_2|^2}{u_1 u_2}} \operatorname{Re}(\vec{A}) \quad , \quad \vec{N}_2 = \sqrt{\frac{|\varphi_1|^2 |\varphi_2|^2}{u_1 u_2}} \operatorname{Im}(\vec{A}) \quad (3.7)$$

where

$$\begin{aligned} u_k &= |\psi_k|^2 + |\varphi_k|^2, \quad k = 1, 2 \\ \vec{A} &= \left[-\frac{\psi_1}{\varphi_1} - \frac{\bar{\psi}_2}{\varphi_2}, i \left(\frac{\psi_1}{\varphi_1} - \frac{\bar{\psi}_2}{\varphi_2} \right), \frac{\psi_1 \bar{\psi}_2}{\varphi_1 \varphi_2} - 1, -i \left(1 + \frac{\psi_1 \bar{\psi}_2}{\varphi_1 \varphi_2} \right) \right] \end{aligned} \quad (3.8)$$

The mean curvature vector defined standardly as

$$\vec{H} = \frac{1}{g_{z\bar{z}}} \vec{X}_{z\bar{z}}$$

is given by

$$\begin{aligned} \vec{H} &= \frac{2}{u_1 u_2} [Re(p\varphi_1\psi_2 + \bar{p}\psi_1\varphi_2), Im(p\varphi_1\psi_2 + \bar{p}\psi_1\varphi_2), \\ &\quad Re(p\varphi_1\bar{\varphi}_2 - \bar{p}\psi_1\bar{\psi}_2), Im(p\bar{\psi}_1\psi_2 - \bar{p}\bar{\varphi}_1\varphi_2)] \end{aligned} \quad (3.9)$$

The components h_1, h_2 of \vec{H} along \vec{N}_1 and \vec{N}_2 (*i.e.* $\vec{H} = h_1\vec{N}_1 + h_2\vec{N}_2$) are

$$h_1 = -\frac{2\text{Re}(p\bar{\varphi}_1\varphi_2)}{\sqrt{|\varphi_1|^2|\varphi_2|^2}u_1u_2}, \quad h_2 = \frac{2\text{Im}(p\bar{\varphi}_1\varphi_2)}{\sqrt{|\varphi_1|^2|\varphi_2|^2}u_1u_2} \quad (3.10)$$

So, the mean curvature $\vec{H}^2 = \sum_{i=1}^4 H^i H^i = h_1^2 + h_2^2$ is equal to

$$\vec{H}^2 = 4 \frac{|p|^2}{u_1 u_2} \quad (3.11)$$

Then the Gaussian curvature is

$$K = -\frac{2}{u_1 u_2} [\log(u_1 u_2)]_{z\bar{z}} \quad (3.12)$$

Finally, the Willmore functional $W = \int \vec{H}^2 [dS]$ is given by

$$W = 4 \int |p|^2 dx dy \quad (3.13)$$

Thus, we have the following

Theorem 3.1 *The generalized Weierstrass formulae*

$$\begin{aligned} X^1 + iX^2 &= \int_{\Gamma} (-\varphi_1\varphi_2 dz' + \psi_1\psi_2 d\bar{z}') , \\ X^1 - iX^2 &= \int_{\Gamma} (\bar{\psi}_1\bar{\psi}_2 dz' - \bar{\varphi}_1\bar{\varphi}_2 d\bar{z}') , \\ X^3 + iX^4 &= \int_{\Gamma} (\varphi_1\bar{\psi}_2 dz' + \psi_1\bar{\varphi}_2 d\bar{z}') , \\ X^3 - iX^4 &= \int_{\Gamma} (\bar{\psi}_1\varphi_2 dz' + \bar{\varphi}_1\psi_2 d\bar{z}') \end{aligned} \quad (3.14)$$

where

$$\begin{aligned} \psi_{1z} &= p\varphi_1 \quad , & \psi_{2z} &= \bar{p}\varphi_2 \quad , \\ \varphi_{1\bar{z}} &= -\bar{p}\psi_1 & \varphi_{2\bar{z}} &= -p\psi_2 \quad , \end{aligned} \quad (3.15)$$

Γ is a contour in \mathbb{C} , define the conformal immersion of a surface into \mathbb{R}^4 . The induced metric is of the form

$$ds^2 = u_1 u_2 dz d\bar{z} \quad (3.16)$$

where $u_k = |\psi_k|^2 + |\varphi_k|^2$ ($k = 1, 2$), the Gaussian and squared mean curvatures are

$$K = -\frac{2}{u_1 u_2} [\log(u_1 u_2)]_{z\bar{z}} \quad , \quad \bar{H}^2 = 4 \frac{|p|^2}{u_1 u_2} \quad . \quad (3.17)$$

The total squared mean curvature (Willmore functional) is given by

$$W = 4 \int |p|^2 dx dy \quad . \quad (3.18)$$

Since the linear system (3.15) contains two arbitrary functions (**Rep** and **Imp**) of two variables, then the formulae (3.14), (3.15) allows us to get any surface in \mathbb{R}^4 . The generalized Weierstrass representation (3.14) defines surfaces in \mathbb{R}^4 up to translations. In the specialized case $\bar{p} = p$ one gets the formulae derived in [35]. In the particular case $\psi_2 = \pm\psi_1$, $\varphi_2 = \pm\varphi_1$, $X_z^4 = X_{\bar{z}}^4 = 0$ and the formulae (3.15)-(3.18) are reduced to those (2.1), (2.2) of the \mathbb{R}^3 case with the substitution $X^1 \leftrightarrow X^2$, $X^3 \leftrightarrow -X^3$.

Note that a linear system of the form (3.1) arises also as the restriction of the Dirac equation to a surface in \mathbb{R}^4 [41].

Note the equations (3.3) can be represented in the form

$$\begin{aligned} d(X^1 + iX^2) &= -\varphi_1 \varphi_2 dz + \psi_1 \psi_2 d\bar{z} \quad , \\ d(X^3 + iX^4) &= \varphi_1 \bar{\psi}_2 dz + \psi_1 \bar{\varphi}_2 d\bar{z} \end{aligned} \quad (3.19)$$

which reveals a symmetry between the pairs of coordinates (X^1, X^2) and (X^3, X^4) .

The formulae (3.3) can be also rewritten in a spinor representation type form

$$d(\sigma_1 X^1 + \sigma_2 X^2 + \sigma_3 X^3 + iIX^4) = V_2^\dagger \begin{pmatrix} 0 & dz \\ d\bar{z} & 0 \end{pmatrix} V_1$$

where

$$V_{1,2} = \begin{pmatrix} \psi_{1,2} & -\overline{\varphi}_{1,2} \\ \varphi_{1,2} & \overline{\psi}_{1,2} \end{pmatrix},$$

σ_i ($i = 1, 2, 3$) are the Pauli matrices and I is the identity matrix.

The condition (3.5), that an immersion is conformal, written as

$$(X_z^1)^2 + (X_z^2)^2 + (X_z^3)^2 + (X_z^4)^2 = 0 \quad (3.20)$$

defines the complex quadric \mathbb{Q}_2

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0 \quad (3.21)$$

in \mathbb{CP}^3 where w_i ($i = 1, 2, 3, 4$) are homogeneous coordinates. A diffeomorphism of \mathbb{Q}_2 to the Grassmannian $\mathbb{G}_{2,4}$ of oriented 2-planes in \mathbb{R}^4 allows us to define the Gauss map $\vec{G}(z)$ for a surface represented by the generalized Weierstrass formulae (3.14). It is given by

$$\vec{G}(z) = \left[1 + \frac{\overline{\psi}_1 \overline{\psi}_2}{\varphi_1 \varphi_2}, i \left(1 - \frac{\overline{\psi}_1 \overline{\psi}_2}{\varphi_1 \varphi_2} \right), i \left(\frac{\overline{\psi}_1}{\varphi_1} + \frac{\overline{\psi}_2}{\varphi_2} \right), \left(\frac{\overline{\psi}_1}{\varphi_1} - \frac{\overline{\psi}_2}{\varphi_2} \right) \right]. \quad (3.22)$$

The Gauss map for surfaces immersed into \mathbb{R}^4 has been studied earlier in the paper [31]. In [31] the Gauss map $\vec{G}(z)$ has been parametrized as follows

$$\vec{G}(z) = [1 + f_1 f_2, i(1 - f_1 f_2), f_1 - f_2, -i(f_1 + f_2)] \quad (3.23)$$

where f_1 and f_2 are complex-valued functions. A surface in \mathbb{R}^4 is then defined by [31]

$$\vec{X} = \int_{\Gamma} \operatorname{Re} \left(\eta \vec{G} dz \right) \quad (3.24)$$

where f_1 and f_2 satisfy the compatibility conditions

$$\operatorname{Im} \left[\left(\frac{f_{1z} \overline{z}}{f_{1\overline{z}}} - 2 \frac{\overline{f}_1 f_{1z}}{1 + |f_1|^2} \right)_{\overline{z}} + \left(\frac{f_{2z} \overline{z}}{f_{2\overline{z}}} - 2 \frac{\overline{f}_2 f_2}{1 + |f_2|^2} \right)_{\overline{z}} \right] = 0 \quad (3.25)$$

and

$$|F_1| = |F_2| \quad (3.26)$$

where $F_i = f_{i\bar{z}} \left(1 + |f_i|^2\right)^{-1}$, $i = 1, 2$. The function η is given by

$$\bar{\eta}^2 = -\frac{4F_1F_2}{H^2 \left(1 + |f_1|^2\right) \left(1 + |f_2|^2\right)} \quad (3.27)$$

where the mean curvature H is expressed via f_1 and f_2 by

$$2(\log H)_z = \frac{f_{1z}\bar{z}}{f_{1\bar{z}}} - 2\frac{\bar{f}_1 f_{1z}}{1 + |f_1|^2} + \frac{f_{2z}\bar{z}}{f_{2\bar{z}}} - 2\frac{\bar{f}_2 f_{2z}}{1 + |f_2|^2} \quad (3.28)$$

Similar to the three-dimensional case this representation includes the complicated compatibility conditions.

Theorem 3.2 *The generalized Weierstrass representation (3.14)-(3.18) and the Gauss map type representation (3.24)-(3.28) are equivalent to each other via the substitution*

$$\eta = i\varphi_1\varphi_2 \quad , \quad f_1 = i\frac{\bar{\psi}_1}{\varphi_1} \quad , \quad f_2 = -i\frac{\bar{\psi}_2}{\varphi_2} \quad (3.29)$$

The proof is straightforward: equations (3.15) and (3.2) give the constraints (3.25)-(3.27) with

$$p = -iF_1\frac{\varphi_1}{\bar{\varphi}_1} \quad , \quad \bar{p} = iF_2\frac{\varphi_2}{\bar{\varphi}_2} \quad (3.30)$$

while (3.14) is converted into (3.24).

4 Generalized Weierstrass representations for surfaces in pseudo-Euclidean spaces

The derivation of the generalized Weierstrass formulae for surfaces immersed into four-dimensional pseudo-Euclidean spaces is rather similar to that of \mathbb{R}^4 .

Theorem 4.1 *The generalized Weierstrass formulae*

$$X^1 + iX^2 = \int_{\Gamma} (\varphi_1\varphi_2 dz' + \psi_1\psi_2 d\bar{z}') \quad ,$$

$$\begin{aligned}
X^1 - iX^2 &= \int_{\Gamma} (\bar{\psi}_1 \bar{\psi}_2 dz' + \bar{\varphi}_1 \bar{\varphi}_2 d\bar{z}') , \\
X^3 + iX^4 &= i \int_{\Gamma} (\bar{\psi}_1 \varphi_2 dz' + \bar{\varphi}_1 \psi_2 d\bar{z}') , \\
X^3 - iX^4 &= -i \int_{\Gamma} (\varphi_1 \bar{\psi}_2 dz' + \psi_1 \bar{\varphi}_2 d\bar{z}')
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
\psi_{1z} &= p\varphi_1 , \quad \psi_{2z} = \bar{p}\varphi_2 , \\
\varphi_{1\bar{z}} &= \bar{p}\psi_1 , \quad \varphi_{2\bar{z}} = p\psi_2 ,
\end{aligned} \tag{4.2}$$

$\psi_{\alpha}, \varphi_{\alpha}, p$ are complex-valued functions, Γ is a contour in \mathbb{C} , define the conformal immersion $\vec{X} : \mathbb{C} \rightarrow \mathbb{R}^{2,2}$ of a surface into the space $\mathbb{R}^{2,2}$. The induced metric is

$$ds^2 = v_1 v_2 dz d\bar{z} \tag{4.3}$$

where $v_{\alpha} = |\psi_{\alpha}|^2 - |\varphi_{\alpha}|^2$, $\alpha = 1, 2$, the Gaussian and mean curvature are of the form

$$K = -\frac{2}{v_1 v_2} [\log(v_1 v_2)]_{z\bar{z}} , \quad \vec{H}^2 = -\frac{4|p|^2}{v_1 v_2} \tag{4.4}$$

and the Willmore functional $W = \int \vec{H}^2 [dS]$ is given by

$$W = -4 \int |p|^2 dx dy . \tag{4.5}$$

The proof is similar to the case of \mathbb{R}^4 , only now equations (4.2) give $(\psi_1 \psi_2)_z = (\varphi_1 \varphi_2)_{\bar{z}}$ and $(\psi_1 \bar{\varphi}_2)_z = (\varphi_1 \bar{\psi}_2)_{\bar{z}}$. The mean curvature vector is given by

$$\begin{aligned}
\vec{H} &= \frac{2}{v_1 v_2} [\text{Re}(p\varphi_1 \psi_2 + \bar{p}\psi_1 \varphi_2), \text{Im}(p\varphi_1 \psi_2 + \bar{p}\psi_1 \varphi_2), \\
&\quad \text{Im}(p\varphi_1 \bar{\varphi}_2 + \bar{p}\psi_1 \bar{\psi}_2), \text{Re}(p\bar{\psi}_1 \psi_2 + \bar{p}\bar{\varphi}_1 \varphi_2)] .
\end{aligned} \tag{4.6}$$

In the particular case $\bar{p} = p$ the formulae (4.1)-(4.5) are reduced to those obtained in [35] with the substitution $X^1 \leftrightarrow X^2$, $X^3 \leftrightarrow X^4$. In contrast to [35] the formulae (4.1)-(4.5) allow to represent an arbitrary surface in $\mathbb{R}^{2,2}$.

Conformal immersions into the Minkowski space $\mathbb{R}^{3,1}$ are given by slightly different formulae.

Theorem 4.2 *The Weierstrass type formulae*

$$\begin{aligned}
X^1 + iX^2 &= \int_{\Gamma} (\varphi_1 \bar{\psi}_2 dz' + \psi_1 \bar{\varphi}_2 d\bar{z}') \quad , \\
X^1 - iX^2 &= \int_{\Gamma} (\bar{\psi}_1 \varphi_2 dz' + \bar{\varphi}_1 \psi_2 d\bar{z}') \quad , \\
X^3 + X^4 &= \int_{\Gamma} (\bar{\psi}_1 \varphi_1 dz' + \psi_1 \bar{\varphi}_1 d\bar{z}') \quad , \\
X^3 - X^4 &= - \int_{\Gamma} (\bar{\psi}_2 \varphi_2 dz' + \psi_2 \bar{\varphi}_2 d\bar{z}') \quad (4.7)
\end{aligned}$$

where

$$\begin{aligned}
\psi_{\alpha z} &= p\varphi_{\alpha} \quad , \\
\varphi_{\alpha \bar{z}} &= q\psi_{\alpha} \quad , \quad \alpha = 1, 2
\end{aligned} \quad (4.8)$$

q and p are real-valued functions, Γ is a contour in \mathbb{C} , define the conformal immersion of a surface into the Minkowski space $\vec{X} : \mathbb{C} \rightarrow \mathbb{R}^{3,1}$. The induced metric on a surface is

$$ds^2 = |\psi_1 \varphi_2 - \varphi_1 \psi_2|^2 dz d\bar{z} \quad , \quad (4.9)$$

the Gaussian curvature is

$$K = - \frac{2}{|\psi_1 \varphi_2 - \psi_2 \varphi_1|^2} [\log (|\psi_1 \varphi_2 - \varphi_1 \psi_2|)]_{z\bar{z}}$$

the squared mean curvature \vec{H}^2 and the Willmore functional are given respectively by

$$\vec{H}^2 = - \frac{4qp}{|\psi_1 \varphi_2 - \psi_2 \varphi_1|^2} \quad , \quad W = -4 \int qp dx dy \quad . \quad (4.10)$$

In this case the linear system (4.8) implies that

$$(\psi_{\alpha} \bar{\varphi}_{\beta})_z = (\varphi_{\alpha} \bar{\psi}_{\beta})_{\bar{z}} \quad \alpha, \beta = 1, 2$$

that guarantee an independence of the *r.h.s.* of (4.7) on the choice of the contour Γ of integration. The rest is straightforward. In particular, the mean curvature vector \vec{H} is of the form

$$\vec{H} = \frac{2}{|\psi_1 \varphi_2 - \varphi_1 \psi_2|^2} [\text{Re} (p\varphi_1 \bar{\varphi}_2 + q\psi_1 \bar{\psi}_2), \text{Re} (ip\bar{\varphi}_1 \varphi_2 + iq\bar{\psi}_1 \psi_2),$$

$$\frac{p}{2} \left(|\varphi_1|^2 - |\varphi_2|^2 \right) + \frac{q}{2} \left(|\psi_1|^2 - |\psi_2|^2 \right), \frac{p}{2} \left(|\varphi_1|^2 + |\varphi_2|^2 \right) + \frac{q}{2} \left(|\psi_1|^2 + |\psi_2|^2 \right) \Big] . \quad (4.11)$$

Since again one has two arbitrary real-valued functions p and q , the Weierstrass type formulae (4.7)-(4.8) allow us to construct any surface immersed into $\mathbb{R}^{3,1}$.

Differential version of all three generalized Weierstrass representations given above can be written in the following common form

$$d \left(\sum_{i=1}^4 \tau_i X^i \right) = \Phi_2^\dagger \begin{pmatrix} 0 & dz \\ d\bar{z} & 0 \end{pmatrix} \Phi_1 \quad (4.12)$$

where \dagger denotes Hermitian conjugation. In the case of immersion into \mathbb{R}^4 one has

$$\tau_1 = \sigma_1 \ , \ \tau_2 = \sigma_2 \ , \ \tau_3 = \sigma_3 \ , \ \tau_4 = i\sigma_4$$

and

$$\Phi_\alpha = \begin{pmatrix} \psi_\alpha & -\overline{\varphi}_\alpha \\ \varphi_\alpha & \overline{\psi}_\alpha \end{pmatrix} \ , \ \alpha = 1, 2 \quad (4.13)$$

where $\sigma_1, \sigma_2, \sigma_3$ are the standard Pauli matrices and σ_4 is an identical 2×2 matrix. At the $\mathbb{R}^{2,2}$ case

$$\tau_1 = \sigma_1 \ , \ \tau_2 = \sigma_2 \ , \ \tau_3 = i\sigma_3 \ , \ \tau_4 = \sigma_4$$

and

$$\Phi_\alpha = \begin{pmatrix} \psi_\alpha & \overline{\varphi}_\alpha \\ \varphi_\alpha & \overline{\psi}_\alpha \end{pmatrix} \ , \ \alpha = 1, 2 \ . \quad (4.14)$$

Finally, the immersion into the Minkowski space $\mathbb{R}^{3,1}$ correspond to

$$\tau_i = \sigma_i \quad (i = 1, 2, 3, 4)$$

and

$$\Phi_1 = \Phi_2 = \begin{pmatrix} \psi_1 & \psi_2 \\ \varphi_2 & \varphi_1 \end{pmatrix} . \quad (4.15)$$

In fact, one can start with the formulae (4.12) to derive the Weierstrass representations in the forms (3.14)-(3.15), (4.1)-(4.2) and (4.7)-(4.8). Indeed, one can show that the 1-form in the *r.h.s.* of (4.12) is closed if the 2×2 matrices Φ_1, Φ_2 obey the Dirac equations

$$\begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix} \Phi_1 = \begin{pmatrix} u & p \\ q & v \end{pmatrix} \Phi_1, \quad \begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix} \Phi_2 = \begin{pmatrix} -\bar{v} & \bar{p} \\ \bar{q} & -\bar{u} \end{pmatrix} \Phi_2 \quad (4.16)$$

where p, q, u, v are arbitrary complex-valued functions. Functions u and v always can be converted to zeros by gauge transformation (redefinition of Φ). Then the reality conditions for X^i are satisfied if matrices Φ_α have the form (4.13), (4.14) or (4.15) while the functions p, q should obey the constraints $p + \bar{q} = 0, p - \bar{q} = 0$ and $p = \bar{p}, q = \bar{q}$, respectively. Consequently, the corresponding formula (4.12) gives rise to the Weierstrass representations considered above.

A formula of the type (4.12) appears naturally [42] in the quaternionic approach to surfaces (see also [38-39,43]) which could provide an invariant formulation of the construction presented above.

In the particular case $\bar{p} = p, \psi_1 = \psi_2 = \psi$ and $\varphi_1 = \varphi_2 = \varphi$ the Weierstrass representation (4.1), (4.2) defines the conformal immersion

$$\begin{aligned} X^1 + iX^2 &= \int_\Gamma (\varphi^2 dz' + \psi^2 d\bar{z}') \\ X^4 &= \int_\Gamma (\bar{\psi}\varphi dz' + \psi\bar{\varphi} d\bar{z}') \end{aligned} \quad (4.17)$$

where

$$\begin{aligned} \psi_z &= p\varphi, \\ \varphi_{\bar{z}} &= p\psi \end{aligned} \quad (4.18)$$

of a surface into the three-dimensional pseudo-Euclidean space with the metric $g_{ik} = \text{diag}(1, 1, -1)$. The induced metric is

$$ds^2 = \left(|\psi|^2 - |\varphi|^2 \right)^2 dz d\bar{z} \quad (4.19)$$

while the squared mean curvature and the Willmore functional are

$$\vec{H}^2 = -\frac{4p^2}{\left(|\psi|^2 - |\varphi|^2 \right)^2}, \quad W = -4 \int p^2 dx dy \quad (4.20)$$

respectively.

So (4.17), (4.18) give the pseudo-Euclidean version of the generalized representation (2.1), (2.2). This representation again generates any surface conformally immersed in $\mathbb{R}^{2,1}$. The representation (4.17), (4.18) for surfaces in $\mathbb{R}^{2,1}$ has been found earlier in [44]. The Kenmotsu type formula for nonminimal surfaces in $\mathbb{R}^{2,1}$ has been given in [45]. An analog of the formulae (2.6)-(2.8) is of the form [45]

$$\vec{X} = \operatorname{Re} \left(\int^z \eta \vec{\phi} dz' \right) \quad (4.21)$$

where

$$\vec{\phi} = [1 + f^2, i(1 - f^2), 2f] \quad (4.22)$$

and

$$2\bar{f}_z = \eta H (1 - |f|^2)^2, \quad (\log H)_z = \frac{1}{f\bar{z}} \left(f_{z\bar{z}} + 2 \frac{\bar{f} f_z f_{\bar{z}}}{1 - |f|^2} \right). \quad (4.23)$$

Comparing the Gauss map vector for the Weierstrass type representation (4.17), (4.18), *i.e.*

$$G(z) = \left[\left(1 + \frac{\varphi^2}{\bar{\psi}^2} \right), i \left(1 - \frac{\varphi^2}{\bar{\psi}^2} \right), 2 \frac{\varphi}{\bar{\psi}} \right] \quad (4.24)$$

with (4.21), (4.22), one concludes that

$$f = \frac{\varphi}{\bar{\psi}}, \quad \eta = \bar{\psi}^2. \quad (4.25)$$

Further, the relation

$$H = \frac{2p}{|\psi|^2 - |\varphi|^2}$$

converts the nonlinear equations (4.23) into the linear system (4.18) and vice versa.

So the Weierstrass representation (4.17), (4.18) and the Kenmotsu type formulae (4.21)-(4.23) are equivalent to each other through the relations (4.25). Obviously, the Kenmotsu type formulae (4.21)-(4.23) can be obtained also as a particular case of the generalized Weierstrass representation for surfaces immersed into $\mathbb{R}^{3,1}$. Indeed, the formulae (4.17)-(4.18) arise from the formulae (4.7)-(4.8) under the reduction $p = q, \bar{\psi}_2 = \varphi_1, \varphi_2 = \bar{\psi}_1$.

5 The Weierstrass representations for time-like surfaces in the 4D pseudo-Euclidean spaces

Hyperbolic (time-like) surfaces with the signature $(+, -)$ appear naturally in the pseudo-Euclidean spaces. They arise as the world-sheets for strings moving in Minkowski space or other pseudo-Euclidean spaces (see [7-9]) and are of interest in differential geometry too [5,6]. For time-like surfaces minimal lines are real. So, to get a surface parametrized by minimal lines, one has to start with the Dirac linear equations which instead of z, \bar{z} contain real independent variables (say ξ and η).

Theorem 5.1 *The formulae*

$$\begin{aligned} X^1 &= \frac{1}{2} \int_{\Gamma} [(\bar{\varphi}_1 \varphi_2 + \varphi_1 \bar{\varphi}_2) d\xi' + (\bar{\psi}_1 \psi_2 + \psi_1 \bar{\psi}_2) d\eta'] \quad , \\ X^2 &= \frac{i}{2} \int_{\Gamma} [(\bar{\varphi}_1 \varphi_2 - \varphi_1 \bar{\varphi}_2) d\xi' + (\bar{\psi}_1 \psi_2 - \psi_1 \bar{\psi}_2) d\eta'] \quad , \\ X^3 &= \frac{1}{2} \int_{\Gamma} [(\varphi_1 \bar{\varphi}_1 - \varphi_2 \bar{\varphi}_2) d\xi' + (\psi_1 \bar{\psi}_1 - \psi_2 \bar{\psi}_2) d\eta'] \quad , \\ X^4 &= \frac{1}{2} \int_{\Gamma} [(\varphi_1 \bar{\varphi}_1 + \varphi_2 \bar{\varphi}_2) d\xi' + (\psi_1 \bar{\psi}_1 + \psi_2 \bar{\psi}_2) d\eta'] \quad , \end{aligned} \tag{5.1}$$

where the functions $\psi_{\alpha}, \varphi_{\alpha}$ ($\alpha = 1, 2$) satisfy the linear system

$$\begin{aligned} \psi_{\alpha\xi} &= p\varphi_{\alpha} \quad , \\ \varphi_{\alpha\eta} &= \bar{p}\psi_{\alpha} \quad , \end{aligned} \quad \alpha = 1, 2 \tag{5.2}$$

Γ is a contour of integration in \mathbb{R}^2 , define an immersion into the Minkowski space $\mathbb{R}^{3,1}$ of a generic time-like surface parametrized by the minimal lines $\xi = \text{const}, \eta = \text{const}$. The induced metric is

$$ds^2 = -|\varphi_1 \psi_2 - \psi_1 \varphi_2|^2 d\eta d\xi \quad , \tag{5.3}$$

the Gaussian and squared mean curvature are given respectively by

$$K = \frac{2}{|\varphi_1 \psi_2 - \psi_1 \varphi_2|^2} [\log(|\varphi_1 \psi_2 - \psi_1 \varphi_2|)]_{z\bar{z}} \quad , \quad \bar{H}^2 = 4 \frac{|p|^2}{|\varphi_1 \psi_2 - \psi_1 \varphi_2|^2} \tag{5.4}$$

while the Willmore functional is

$$W = 2 \int |p|^2 d\eta d\xi \quad . \quad (5.5)$$

The formulae (5.1), (5.2) allow to construct any surface in $\mathbb{R}^{3,1}$. The mean curvature vector defined as

$$\vec{H} = \frac{1}{g_{\eta\xi}} \vec{X}_{\eta\xi}$$

is of the form

$$\begin{aligned} \vec{H} = & -\frac{|\varphi_1\psi_2 - \psi_1\varphi_2|^{-2}}{2} \text{Re}[p(\bar{\psi}_1\varphi_2 + \bar{\psi}_2\varphi_1), ip(\bar{\psi}_1\varphi_2 + \bar{\psi}_2\varphi_1), \\ & p(\bar{\psi}_1\varphi_1 - \bar{\psi}_2\varphi_2), p(\bar{\psi}_1\varphi_1 + \bar{\psi}_2\varphi_2)] \quad . \end{aligned} \quad (5.6)$$

It is interesting that the formulae similar to (5.1) have appeared in completely different context as parametrization of constraint for coordinates of classical string governed by the Nambu-Goto Lagrangian [46]. The formulae from [46] look like

$$\begin{aligned} X_\xi^1 &= ac + bd \quad , \\ X_\xi^2 &= ad - bc \quad , \\ (X^4 + X^3)_\xi &= a^2 + b^2 \quad , \\ (X^4 - X^3)_\xi &= c^2 + d^2 \end{aligned} \quad (5.7)$$

where a, b, c, d are real-valued function; similar formulae with real-valued functions $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ hold for X_η^i . It is not difficult to see that (5.7) and (5.1) coincide under the identification

$$\begin{aligned} \varphi_1 &= a + ib \quad , & \varphi_2 &= c + id \\ \psi_1 &= \tilde{a} + i\tilde{b} \quad , & \psi_2 &= \tilde{c} + i\tilde{d} \end{aligned} \quad (5.8)$$

Note that the paper [46] contains no the linear system (5.2) and formulae (5.3)-(5.6). The result of the paper [46] suggest possible applications of the representation (5.1), (5.2) in the theory of classical strings. For example, the Nambu-Goto action takes the form

$$S_{NG} = \frac{\alpha}{2} \int |\varphi_1\psi_2 - \psi_1\varphi_2|^2 d\eta d\xi \quad .$$

Theorem 5.2 *The Weierstrass type formulae*

$$\begin{aligned}
X^1 &= \frac{1}{2} \int_{\Gamma} [(\varphi_1 \tilde{\varphi}_2 + \tilde{\varphi}_1 \varphi_2) d\xi' + (\psi_1 \tilde{\psi}_2 + \tilde{\psi}_1 \psi_2) d\eta'] \quad , \\
X^2 &= \frac{1}{2} \int_{\Gamma} [(\varphi_1 \tilde{\varphi}_1 - \tilde{\varphi}_2 \varphi_2) d\xi' + (\psi_1 \tilde{\psi}_1 - \tilde{\psi}_2 \psi_2) d\eta'] \quad , \\
X^3 &= \frac{1}{2} \int_{\Gamma} [(\tilde{\varphi}_1 \varphi_2 - \varphi_1 \tilde{\varphi}_2) d\xi' + (\tilde{\psi}_1 \psi_2 - \psi_1 \tilde{\psi}_2) d\eta'] \quad , \\
X^4 &= \frac{1}{2} \int_{\Gamma} [(\varphi_1 \tilde{\varphi}_1 + \varphi_2 \tilde{\varphi}_2) d\xi' + (\psi_1 \tilde{\psi}_1 + \psi_2 \tilde{\psi}_2) d\eta']
\end{aligned} \tag{5.9}$$

where

$$\begin{aligned}
\psi_{\alpha\xi} &= p\varphi_{\alpha} \quad , & \tilde{\psi}_{\alpha\xi} &= q\tilde{\varphi}_{\alpha} \quad , \\
\varphi_{\alpha\eta} &= q\psi_{\alpha} \quad , & \tilde{\varphi}_{\alpha\eta} &= p\tilde{\psi}_{\alpha} \quad ,
\end{aligned} \quad (\alpha = 1, 2) \tag{5.10}$$

$\psi_{\alpha}, \varphi_{\alpha}, \tilde{\psi}_{\alpha}, \tilde{\varphi}_{\alpha}$ ($\alpha = 1, 2$) and p, q are real-valued functions, Γ is a contour of integration in \mathbb{R}^2 , define an immersion into $\mathbb{R}^{2,2}$ of a generic surface parametrized by minimal lines. The induced metric is

$$ds^2 = (\psi_1 \varphi_2 - \psi_2 \varphi_1)(\tilde{\varphi}_1 \tilde{\psi}_2 - \tilde{\psi}_1 \tilde{\varphi}_2) d\xi d\eta \tag{5.11}$$

while the Gaussian and the squared mean curvatures are

$$K = -\frac{2}{(\psi_1 \varphi_2 - \psi_2 \varphi_1)(\tilde{\varphi}_1 \tilde{\psi}_2 - \tilde{\psi}_1 \tilde{\varphi}_2)} \left\{ \log \left[(\psi_1 \varphi_2 - \psi_2 \varphi_1)(\tilde{\varphi}_1 \tilde{\psi}_2 - \tilde{\psi}_1 \tilde{\varphi}_2) \right] \right\}_{z\bar{z}} \tag{5.12}$$

$$\vec{H}^2 = \frac{4qp}{(\psi_1 \varphi_2 - \psi_2 \varphi_1)(\tilde{\psi}_1 \tilde{\varphi}_2 - \tilde{\psi}_2 \tilde{\varphi}_1)} \tag{5.13}$$

and the Willmore functional is given by

$$W = -2\epsilon \int qp d\xi d\eta \quad . \tag{5.14}$$

where $\epsilon = \text{sign} \left[(\psi_1 \varphi_2 - \psi_2 \varphi_1)(\tilde{\varphi}_1 \tilde{\psi}_2 - \tilde{\psi}_1 \tilde{\varphi}_2) \right]$.

In this case one has from (5.10)

$$\left(\psi_{\alpha} \tilde{\psi}_{\beta} \right)_{\xi} = (\varphi_{\alpha} \tilde{\varphi}_{\beta})_{\eta} \quad , \quad \alpha, \beta = 1, 2$$

that gives rise to (5.9). The mean curvature vector is

$$\begin{aligned}
\vec{H} &= \frac{1}{g_{\eta\xi}} \vec{X}_{\eta\xi} = \left[(\psi_1 \varphi_2 - \psi_2 \varphi_1)(\tilde{\varphi}_1 \tilde{\psi}_2 - \tilde{\psi}_1 \tilde{\varphi}_2) \right]^{-1} [(q\psi_1 \tilde{\varphi}_2 + p\varphi_1 \tilde{\psi}_2) + (1 \leftrightarrow 2), \\
&\quad (q\psi_1 \tilde{\varphi}_1 + p\varphi_1 \tilde{\psi}_1) - (1 \leftrightarrow 2), (q\psi_2 \tilde{\varphi}_1 + p\varphi_2 \tilde{\psi}_1) - (1 \leftrightarrow 2), (q\psi_1 \tilde{\varphi}_1 + p\varphi_1 \tilde{\psi}_1) + (1 \leftrightarrow 2)].
\end{aligned}$$

For time-like surfaces an analog of the formula (4.12) looks like

$$d\left(\sum \tau_i X^i\right) = \Phi_2^\dagger \begin{pmatrix} 0 & d\xi \\ d\eta & 0 \end{pmatrix} \Phi_1$$

where for the case of the space $\mathbb{R}^{3,1}$ one has

$$\tau_1 = \sigma_1 \quad , \quad \tau_2 = i\sigma_2 \quad , \quad \tau_3 = \sigma_3 \quad , \quad \tau_4 = \sigma_4$$

and

$$\Phi_1 = \Phi_2 = \begin{pmatrix} \psi_1 & \psi_2 \\ \varphi_1 & \varphi_2 \end{pmatrix}$$

while for the space $\mathbb{R}^{2,2}$

$$\tau_1 = \sigma_1 \quad , \quad \tau_2 = \sigma_3 \quad , \quad \tau_3 = i\sigma_2 \quad , \quad \tau_4 = \sigma_4$$

and

$$\Phi_1 = \begin{pmatrix} \psi_1 & \psi_2 \\ \varphi_1 & \varphi_2 \end{pmatrix} \quad , \quad \Phi_2 = \begin{pmatrix} \tilde{\varphi}_1 & \tilde{\varphi}_2 \\ \tilde{\psi}_1 & \tilde{\psi}_2 \end{pmatrix} \quad .$$

Note that in the special case $\psi_1 = \psi_2$, $\varphi_1 = \varphi_2$ (for (5.1)-(5.2)) and at $\psi_1 = \psi_2$, $\varphi_1 = \varphi_2$, $\tilde{\psi}_1 = \tilde{\psi}_2$, $\tilde{\varphi}_1 = \tilde{\varphi}_2$ (for (5.9)-(5.10)) the immersions (5.1)-(5.2) and (5.9)-(5.10) are reduced to maps $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ ($(\xi, \eta) \rightarrow (X^1, X^4)$ or $(\xi, \eta) \rightarrow (X^1, X^2)$).

It is interesting that the generalized Weierstrass formulae given in the theorems (3.1), (4.1), (4.2), (5.1), (5.2) take place also in the case when the quantities ψ_α and φ_α are elements of the Grassmannian algebra, *i.e.* when they anticommute to each other. The geometric characteristics of surfaces are given by formulae similar to those of the theorems (3.1), (4.1), (4.2), (5.1), (5.2). This type of the Weierstrass representation is of the interest in the string theory.

6 Surfaces in four-dimensional Riemann space

The results of the previous sections apparently can be extended to the case of immersion into generic four-dimensional Riemann space with the metric tensor

g_{ik} .

Theorem 6.1 *The generalized Weierstrass formulae define an immersion of surface into the four dimensional Riemann space with the metric tensor g_{ik} . The induced metric is*

$$ds^2 = g_{\xi\xi} d\xi^2 + 2g_{\xi\eta} d\xi d\eta + g_{\eta\eta} d\eta^2 \quad (6.1)$$

where

$$g_{\xi\xi} = g_{ik} X_\xi^i X_\xi^k, \quad g_{\xi\eta} = g_{ik} X_\xi^i X_\eta^k, \quad g_{\eta\eta} = g_{ik} X_\eta^i X_\eta^k. \quad (6.2)$$

The Gaussian and mean curvature are calculated straightforwardly. One can choose any of the formulae presented above to define the coordinates X^i via the solutions of the linear system. It is obvious, however, that to get an immersion which is, for instance, locally conformal around a point one should choose the Weierstrass type formula for the 4D pseudo-Euclidean space with the metric which coincide with the signature of the desired 4D Riemann space.

In the case of conformally-Euclidean spaces ($g_{ik} = e^{2\sigma} \delta_{ik}$, $i, k = 1, 2, 3, 4$, σ is a function and δ_{ik} is the Kronecker symbol) the immersion is the conformal one:

$$ds^2 = e^{2\sigma} u_1 u_2 dz d\bar{z}. \quad (6.3)$$

The Gaussian and mean curvatures are

$$K = -2e^{-2\sigma} \frac{[2\sigma + \log(u_1 u_2)]_{z\bar{z}}}{u_1 u_2}, \quad H^2 = 4e^{-2\sigma} \frac{|p|^2}{u_1 u_2}. \quad (6.4)$$

For the Willmore functional one gets

$$W = 4 \int |p|^2 dx dy. \quad (6.5)$$

A special case of immersions into the space \mathbb{S}^4 of constant curvature had attracted recently the particular interest (see *e.g.* [4,47]). To describe it we choose the Riemann form for the metric of \mathbb{S}^4 , *i.e.* (see *e.g.* [48])

$$e^{2\sigma} = \left[1 + \frac{K_0}{4} \sum_{i=1}^4 (X^i)^2 \right]^{-2} \quad (6.6)$$

where K_0 is the curvature. Then the formulae (3.14), (3.15) define the conformal immersion of a surface into \mathbb{S}^4 . At $\psi_2 = \pm\psi_1$, $\varphi_2 = \pm\varphi_1$ ($X^4 = \text{const}$) one has the conformal immersion into \mathbb{S}^3 . The generalized Weierstrass representation provides us an effective method to study immersions into \mathbb{S}^3 and \mathbb{S}^4 and, consequently, the Willmore surfaces in \mathbb{R}^3 .

Similarly, one can get conformal immersions into hyperbolic spaces $\mathbb{S}^{3,1}$ and $\mathbb{S}^{2,2}$ with the constant curvature. Indeed, one has simply to take

$$e^{2\sigma} = \left[1 + \frac{K_0}{4} \overline{X}^2 \right]^{-2} . \quad (6.7)$$

7 Integrable deformations

In construction of deformations of surfaces given by the Weierstrass representations we follow to the general approach of [12-13].

So we assume that all quantities in the linear problems (3.15), (4.2), (4.8), (5.2), (5.10) (except z , \bar{z} , ξ and η) depend on the new deformations parameters t_n . Then we assume that this dependence on t_n is such that there are operators A_n , B_n , C_n , D_n such that equations

$$\begin{aligned} \psi_{t_n} &= A_n \psi + B_n \varphi \quad , \\ \varphi_{t_n} &= A_n \psi + B_n \varphi \end{aligned} \quad , \quad n = 1, 2, 3, \dots \quad (7.1)$$

hold. The compatibility conditions of (3.15), (4.2), (4.8), (5.2), (5.10) with (7.1) fix the dependence of ψ , φ and p , q on t_n and, consequently, define the deformations of surfaces. Concrete cases are governed by different specializations (reductions) of the DS hierarchy.

Let us consider first immersions of space-like surfaces into the Minkowski space $\mathbb{R}^{3,1}$. In this case p and q are real-valued functions. The corresponding deformations are generated by the "real" DSII hierarchy (with real-valued p and q). In particular, in equations (A.2), (A.5) the constants α , γ , α_2 are real. Since (A.11) is obviously the integral of motion also for real-valued p and q ,

then, in virtue of (4.10), the Willmore functional W remains invariant under these DSII deformations.

For the Weierstrass representations of the space-like surfaces in \mathbb{R}^4 and $\mathbb{R}^{2,2}$ we have the linear system (A.1) with the following reductions:

$$\begin{pmatrix} 0 & p_1 \\ q_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & p \\ \varepsilon \bar{p} & 0 \end{pmatrix} \quad \text{for } \Phi_1 \quad (7.2)$$

and

$$\begin{pmatrix} 0 & p_2 \\ q_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \bar{p} \\ \varepsilon p & 0 \end{pmatrix} \quad \text{for } \Phi_2 \quad (7.3)$$

where $\varepsilon = -1$ for \mathbb{R}^4 and $\varepsilon = 1$ for $\mathbb{R}^{2,2}$. Both the reductions (7.2) and (7.3) are admissible by all equations of the DSII hierarchy if one chooses A_n and D_n as in (A.12).

In our case we have different linear problems for ψ_1, φ_1 and ψ_2, φ_2 . To have the same equations for p it is enough to take

$$\begin{aligned} A_{2n-1} &= \partial_{\bar{z}}^{2n-1} + \dots, & D_{2n-1} &= \partial_z^{2n-1} + \dots, \\ A_{2n} &= i\partial_{\bar{z}}^{2n} + \dots, & D_{2n} &= -i\partial_z^{2n} + \dots \end{aligned} \quad (7.4)$$

in equations for ψ_1, φ_1 and

$$\begin{aligned} A_{2n-1} &= \partial_{\bar{z}}^{2n-1} + \dots, & D_{2n-1} &= \partial_z^{2n-1} + \dots, \\ A_{2n} &= -i\partial_{\bar{z}}^{2n} + \dots, & D_{2n} &= i\partial_z^{2n} + \dots \end{aligned} \quad (7.5)$$

in the case of ψ_2, φ_2 . In particular, one gets

$$\begin{aligned} p_{t_2} &= i(p_z z + p_{\bar{z}\bar{z}} + up) \quad , \\ u_{z\bar{z}} &= -2\varepsilon|p|_{zz}^2 - 2\varepsilon|p|_{\bar{z}\bar{z}}^2 \end{aligned} \quad (7.6)$$

and

$$\psi_{1t_2} = i(\partial_{\bar{z}}^2 + w_1)\psi_1 + i(p_z - p\partial_z)\varphi_1 \quad , \quad (7.7)$$

$$\varphi_{1t_2} = -i\varepsilon(\bar{p}_{\bar{z}} - \bar{p}\partial_{\bar{z}})\psi_1 - i(\partial_z^2 + w_2)\varphi_1$$

while

$$\psi_{2t_2} = -i(\partial_{\bar{z}}^2 + w_1)\psi_2 - i(\bar{p}_z - \bar{p}\partial_z)\varphi_2 \quad , \quad (7.8)$$

$$\varphi_{2t_2} = i\varepsilon(p_{\bar{z}} - p\partial_{\bar{z}})\psi_2 + i(\partial_z^2 + w_2)\varphi_2$$

where $w_{1z} = -2\varepsilon|p|_{\bar{z}}^2$, $w_{2\bar{z}} = 2\varepsilon|p|_z^2$.

The t_3 deformation is given now by equation (A.15) and the deformations of ψ_1, φ_1 and ψ_2, φ_2 are given by (7.1), (A.8) with the reduction (7.2) and (7.3), respectively.

Thus, in the cases of \mathbb{R}^4 and $\mathbb{R}^{2,2}$ deformations of surfaces are generated by the proper DSII equation (7.6) and the corresponding hierarchy. Properties of solutions of the DSII equation (7.6) are essentially different for different signs of ε . Consequently, the properties of deformations of surfaces in \mathbb{R}^4 and $\mathbb{R}^{2,2}$ will differ too.

In both cases $C_1 = \int |p|^2 dx dy$ is the integral of motion for the whole hierarchy. Hence, the Willmore functionals for surfaces immersed into \mathbb{R}^4 and $\mathbb{R}^{2,2}$ are invariant under deformations generated by the DSII hierarchy.

For the time-like surfaces the linear problem (5.2), (5.10), obviously, give rise to the DSI hierarchy. In the case of the time-like surfaces in $\mathbb{R}^{3,1}$ the reduction is $q = \bar{p}$, and, consequently, the deformations of surfaces given by (5.1), (5.2) are generated by the DSI hierarchy under the reduction $q = \bar{p}$. In particular, one has equations (A.13)-(A.15) with $\varepsilon = 1$. Again, the Willmore functional $W = 2 \int |p|^2 d\xi d\eta$ is invariant under all these deformations.

Time-like surfaces in $\mathbb{R}^{2,2}$ are associated with the two different linear problems (5.10). To have the same evolution equations for the pair p and q , one has to choose the deformations of $\psi, \varphi, \tilde{\psi}, \tilde{\varphi}$ in the following form

$$\begin{pmatrix} \psi_\alpha \\ \varphi_\alpha \end{pmatrix}_{t_n} = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \begin{pmatrix} \psi_\alpha \\ \varphi_\alpha \end{pmatrix} \quad \alpha = 1, 2 \quad , \quad (7.9)$$

$$\begin{pmatrix} \tilde{\psi}_\alpha \\ \tilde{\varphi}_\alpha \end{pmatrix}_{t_n} = \begin{pmatrix} \tilde{A}_n & \tilde{B}_n \\ \tilde{C}_n & \tilde{D}_n \end{pmatrix} \begin{pmatrix} \tilde{\psi}_\alpha \\ \tilde{\varphi}_\alpha \end{pmatrix} \quad \alpha = 1, 2 \quad , \quad (7.10)$$

where $A_n, B_n, C_n, D_n, \tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n$ are differential operators with real-valued coefficients and

$$\tilde{A}_n = (-1)^{n-1} A_n[p \leftrightarrow q] \quad , \quad \tilde{B}_n = (-1)^{n-1} B_n[p \leftrightarrow q] \quad ,$$

$$\tilde{C}_n = (-1)^{n-1} C_n[p \leftrightarrow q] \quad , \quad \tilde{D}_n = (-1)^{n-1} D_n[p \leftrightarrow q] \quad .$$

For example, the t_2 -flow for $\psi_\alpha, \varphi_\alpha$ is given by (A.2), (A.6) with real α_2 while for $\tilde{\psi}_\alpha, \tilde{\varphi}_\alpha$ it is given by (7.10) with the opposite signs of α_2 and substitution $p \leftrightarrow q$. In both cases one has equation (A.5). The Willmore functional $W = 2 \int q p d\xi d\eta$ is clearly an invariant of all these deformations.

Thus, though the deformations for surfaces in $\mathbb{R}^4, \mathbb{R}^{3,1}$ and $\mathbb{R}^{2,2}$ are governed by different nonlinear integrable equations, they have the following common property.

Theorem 7.1 *The DSII hierarchy generates integrable deformations of space-like surfaces immersed into $\mathbb{R}^4, \mathbb{R}^{3,1}$ and $\mathbb{R}^{2,2}$ via the generalized Weierstrass representations. The DSI hierarchy generates integrable deformations of time-like surfaces in $\mathbb{R}^{3,1}$ and $\mathbb{R}^{2,2}$. In all cases the Willmore functionals W for surfaces are invariant under the corresponding deformations ($W_{t_n} = 0$).*

DS hierarchy of integrable equations is well studied [49]-[51]. This provides us a broad class of deformations of surfaces in $\mathbb{R}^4, \mathbb{R}^{3,1}$ and $\mathbb{R}^{2,2}$ given explicitly. Moreover, since the inverse spectral transform method allows us to linearize the initial-value problem

$$p(z, \bar{z}, t_n = 0), q(z, \bar{z}, t_n = 0) \rightarrow p(z, \bar{z}, t_n), q(z, \bar{z}, t_n)$$

for soliton equations of the DS hierarchy (see *e.g.* [49]-[51]), then the generalized Weierstrass formulae allows us to linearize the initial-value problem for the deformation of surfaces $\vec{X}(z, \bar{z}, 0) \rightarrow \vec{X}(z, \bar{z}, t_n)$. In virtue of all that, the deformations generated by the DS hierarchy can be referred as integrable one.

Higher integrals of motion for the DS hierarchy are also certain functionals on surfaces invariant under deformations generated by the DS hierarchy. Since the Willmore functional W is invariant under the conformal transformations in four-dimensional spaces, then it is quite natural to suggest

Conjecture 7.1 *Higher integrals of motion for the DS hierarchy are functionals on surfaces in \mathbb{R}^4 , $\mathbb{R}^{3,1}$ and $\mathbb{R}^{2,2}$ which are invariant under conformal transformations in these spaces.*

For tori in \mathbb{R}^3 an analogous conjecture has been proved in [17].

Deformations of ψ , φ , p allows us to find the deformation equations for coordinates \vec{X} and other geometrical quantities. In the case of the space \mathbb{R}^4 , using (3.14), (7.6), (7.7), (7.8), one obtains

$$\begin{aligned} X_{1t_2} &= \frac{1}{2i} \left[\bar{\psi}_2 \overleftrightarrow{\partial} \bar{\psi}_1 - \varphi_2 \overleftrightarrow{\partial} \varphi_1 - c.c. \right] , \\ X_{2t_2} &= \frac{1}{2} \left[\bar{\psi}_2 \overleftrightarrow{\partial} \bar{\psi}_1 + \varphi_2 \overleftrightarrow{\partial} \varphi_1 + c.c. \right] , \\ X_{3t_2} &= \frac{1}{2i} \left[\varphi_2 \overleftrightarrow{\partial} \bar{\psi}_1 + \bar{\psi}_2 \overleftrightarrow{\partial} \varphi_1 - c.c. \right] + I_1 , \\ X_{4t_2} &= \frac{1}{2} \left[\varphi_2 \overleftrightarrow{\partial} \bar{\psi}_1 - \bar{\psi}_2 \overleftrightarrow{\partial} \varphi_1 + c.c. \right] + I_2 , \end{aligned} \quad (7.11)$$

where $f \overleftrightarrow{\partial} g = f \partial g - g \partial f$ and

$$\begin{aligned} I_1 &= \frac{1}{2i} \int_{\Gamma} [(\bar{w}_1 - w_2) (\bar{\psi}_1 \varphi_2 - \varphi_1 \bar{\psi}_2) dz - c.c.] , \\ I_2 &= \frac{1}{2} \int_{\Gamma} [(\bar{w}_1 - w_2) (\bar{\psi}_1 \varphi_2 + \varphi_1 \bar{\psi}_2) dz + c.c.] . \end{aligned}$$

The deformations (7.11) of the coordinates can be decomposed into the normal and tangential parts

$$\vec{X}_{t_2} = a \vec{N}_1 + b \vec{N}_2 + c \vec{X}_z + \bar{c} \vec{X}_{\bar{z}} . \quad (7.12)$$

Using (3.3), (3.7), (3.8) and (7.11), one gets

$$\begin{aligned} a &= \frac{i}{2} \left(|\varphi_1|^2 |\varphi_2|^2 u_1 u_2 \right)^{-1/2} \times \\ &\quad \times \left\{ \left[u_1 \left(\varphi_1 \bar{\psi}_1 \bar{\varphi}_2 \bar{\partial} \bar{\varphi}_1 + |\varphi_2|^2 \varphi_1 \partial \bar{\psi}_2 \right) - c.c. \right] - 1 \leftrightarrow 2 \right\} + \Delta_a , \\ b &= \frac{1}{2} \left(|\varphi_1|^2 |\varphi_2|^2 u_1 u_2 \right)^{-1/2} \times \\ &\quad \times \left\{ \left[u_1 \left(\varphi_2 \bar{\psi}_2 \bar{\varphi}_1 \bar{\partial} \bar{\varphi}_2 - |\varphi_2|^2 \varphi_1 \partial \bar{\psi}_2 \right) + c.c. \right] + 1 \leftrightarrow 2 \right\} + \Delta_b , \\ c &= \frac{i}{2} (u_1 u_2)^{-1} \{ u_1 u_{2z} - u_2 u_{1z} \} + \Delta_c \end{aligned}$$

where

$$\begin{aligned}
\Delta_a &= \left(|\varphi_1|^2 |\varphi_2|^2 u_1 u_2 \right)^{-1/2} \times \\
&\quad \times \left[I_1 \left(\operatorname{Re}(\psi_1 \varphi_1 \bar{\psi}_2 \bar{\varphi}_2) - |\varphi_1|^2 |\varphi_2|^2 \right) + I_2 \operatorname{Im}(\psi_1 \varphi_1 \bar{\psi}_2 \bar{\varphi}_2) \right] , \\
\Delta_b &= \left(|\varphi_1|^2 |\varphi_2|^2 u_1 u_2 \right)^{-1/2} \times \\
&\quad \times \left[I_1 \operatorname{Im}(\psi_1 \varphi_1 \bar{\psi}_2 \bar{\varphi}_2) - I_2 \left(\operatorname{Re}(\psi_1 \varphi_1 \bar{\psi}_2 \bar{\varphi}_2) + |\varphi_1|^2 |\varphi_2|^2 \right) \right] , \\
\Delta_c &= \frac{1}{2} (u_1 u_2)^{-1} [I_1 (\psi_1 \bar{\varphi}_1 + \psi_2 \bar{\varphi}_2) - i I_2 (\psi_1 \bar{\varphi}_1 - \psi_2 \bar{\varphi}_2)] .
\end{aligned}$$

8 Explicit deformations of surfaces

All explicit solutions of the DS hierarchy provide us deformations of surfaces given by explicit formulae. We will present here two classes of deformations for the space-like surfaces immersed in \mathbb{R}^4 and $\mathbb{R}^{2,2}$ generated by the two basic classes of solutions of the DSII hierarchy. We will give only final formulae omitting all calculations which can be found in [49-54].

The first class is given by solutions which are parametrized by arbitrary functions of a single variable. The corresponding ψ_α , φ_α and p are given by [50,13]

$$\begin{aligned}
\psi^{(\alpha)} &= e^{-\lambda \bar{z} + \sum_{n=2} \lambda^n t_n} + \\
&\quad + \frac{1}{2\pi i} \int \int_{\mathbb{C}} \frac{d\mu \wedge d\bar{\mu}}{\mu - \lambda} \sum_{k,m=1}^N \left[\xi_k^{(\alpha)} (1 + A^{(\alpha)})_{km}^{-1} g_m^{(\alpha)}(\mu) \Gamma^{-1}(\mu) \Gamma(\lambda) \right]_{11} , \\
\varphi^{(\alpha)} &= \frac{1}{2\pi i} \int \int_{\mathbb{C}} \frac{d\mu \wedge d\bar{\mu}}{\mu - \lambda} \sum_{k,m=1}^N \left[\xi_k^{(\alpha)} (1 + A^{(\alpha)})_{km}^{-1} g_m^{(\alpha)}(\mu) \Gamma^{-1}(\mu) \Gamma(\lambda) \right]_{21} , \\
p &= \frac{1}{2\pi i} \int \int_{\mathbb{C}} \frac{d\mu \wedge d\bar{\mu}}{\mu} \sum_{k,m=1}^N \left[\xi_k^{(1)} (1 + A^{(1)})_{km}^{-1} g_m^{(1)}(\mu) \Gamma^{-1}(\mu) \right]_{12} ,
\end{aligned} \tag{8.1}$$

where $\alpha = 1, 2$,

$$A_{km}(z, \bar{z}) = \frac{1}{2\pi i} \int \int_{\mathbb{C}} d\lambda \wedge d\bar{\lambda} \int \int_{\mathbb{C}} \frac{d\mu \wedge d\bar{\mu}}{\mu - \lambda} g_k^{(\alpha)}(\mu) \Gamma^{-1}(\mu) \Gamma(\lambda) f_m^{(\alpha)}(\lambda) \tag{8.2}$$

$$f_k^{(\alpha)}(\lambda, \bar{\lambda}) = \begin{pmatrix} f_{1k}^{(\alpha)}(\lambda, \bar{\lambda}) & \overline{\varepsilon f_{2k}^{(\alpha)}(-\lambda, -\bar{\lambda})} \\ g_{2k}^{(\alpha)}(\lambda, \bar{\lambda}) & \overline{f_{1k}^{(\alpha)}(-\lambda, -\bar{\lambda})} \end{pmatrix}, \quad (8.3)$$

$$g_k^{(\alpha)}(\lambda, \bar{\lambda}) = \begin{pmatrix} g_{1k}^{(\alpha)}(\lambda, \bar{\lambda}) & \overline{\varepsilon f_{2k}^{(\alpha)}(-\bar{\lambda}, -\lambda)} \\ g_{2k}^{(\alpha)}(\lambda, \bar{\lambda}) & \overline{g_{1k}^{(\alpha)}(-\bar{\lambda}, -\lambda)} \end{pmatrix},$$

$$\xi_m^{(\alpha)}(z, \bar{z}) = \int \int_{\mathbb{C}} d\lambda \wedge d\bar{\lambda} \Gamma(\lambda) f_m^{(\alpha)}(\lambda), \quad (8.4)$$

$$\eta_k^{(\alpha)}(z, \bar{z}) = \frac{1}{2\pi i} \int \int_{\mathbb{C}} d\mu \wedge d\bar{\mu} g_k^{(\alpha)}(\mu) \Gamma^{-1}(\mu) \quad (8.5)$$

and

$$\Gamma(\lambda) = \begin{pmatrix} e^{-\lambda \bar{z} + \sum_{n=2} \lambda^n t_n} & 0 \\ 0 & e^{\lambda z - \sum_{n=2} \lambda^n t_n} \end{pmatrix}$$

and

$$f_k^{(2)} = \overline{f}_k^{(1)}, \quad g_k^{(2)} = \overline{g}_k^{(1)}.$$

The 2×2 matrices ξ_m and η_k are, in fact, of the form

$$\xi_m^{(\alpha)} = \begin{pmatrix} \overline{\xi}_{m1}^{(\alpha)}(z) & \xi_{m2}^{(\alpha)}(z) \\ \varepsilon \overline{\xi}_{m2}^{(\alpha)}(z) & \xi_{m1}^{(\alpha)}(z) \end{pmatrix}, \quad \eta_m^{(\alpha)} = \begin{pmatrix} \overline{\eta}_{k1}^{(\alpha)}(z) & \eta_{k2}^{(\alpha)}(z) \\ \varepsilon \overline{\eta}_{k2}^{(\alpha)}(z) & \eta_{k1}^{(\alpha)}(z) \end{pmatrix} \quad (8.6)$$

where $\xi_{m\beta}^{(\alpha)}(z)$, $\eta_{k\beta}^{(\alpha)}(z)$ ($m, k = 1, \dots$, $\alpha, \beta = 1, 2$) are arbitrary holomorphic functions. They are, in essence, the complex Fourier transform of the functions $f_{\alpha k}(-\bar{\lambda})$ and $g_{\alpha k}(-\bar{\lambda})$. So the solution (8.1) is parametrized by $4N$ arbitrary holomorphic functions.

Consequently, the generalized Weierstrass formulae (3.14) give us a family of immersed surfaces parametrized by $4N$ arbitrary holomorphic functions. Varying the times t_n , one gets integrable deformations of these surfaces.

The multisoliton solutions form the second class of exact solutions. In this

case [52]

$$p = -2\varepsilon \sum_{i=1}^N \overline{A_{2i}}(z) e^{-\overline{\lambda_i} z}$$

$$\overline{\psi}_1 = e^{\lambda z} + \sum_{i=1}^N \frac{A_{1i}(z)}{\lambda_i - \lambda} e^{(\lambda - \lambda_i) z} \quad (8.7)$$

$$\overline{\varphi}_1 = \sum_{i=1}^N \frac{\overline{A_{2i}}(z)}{\lambda_i - \lambda} e^{(\lambda - \lambda_i) z}$$

where the column $\mathbf{A}_i = \begin{pmatrix} A_{1i} \\ A_{2i} \end{pmatrix}$ obey the system of equations

$$\begin{pmatrix} e^{\lambda_n z} \\ 0 \end{pmatrix} - \sum_{j=1, j \neq n}^N \frac{\mathbf{A}_j}{\lambda_n - \lambda_j} e^{(\lambda_n - \lambda_j) z} = (z + \mu_n) \mathbf{A}_n + \nu_n \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \overline{\mathbf{A}}_n, \quad n = 1, \dots, N, \quad (8.8)$$

where λ_i, μ_n, ν_n are arbitrary complex constants. For ψ_2 and φ_2 one has similar expressions with the substitution $\lambda_i \rightarrow \tilde{\lambda}_i, \mu_n \rightarrow \tilde{\mu}_n, \nu_n \rightarrow \tilde{\nu}_n$ which should be chosen so that

$$\sum_{i=1}^n \overline{A_{2i}} \exp(-\overline{\lambda_i} z) = \sum_{i=1}^n A_{2i} \exp(-\lambda_i z). \quad (8.9)$$

This is the condition that from the linear problem for ψ_2, φ_2 one gets \overline{p} instead of p at the problem for ψ_1, φ_1 . It can be shown [52] that

$$|p|^2 = -4\varepsilon [\log \det D]_z \overline{z} \quad (8.10)$$

where the $2N \times 2N$ matrix D is given by

$$D = \begin{pmatrix} M & Q \\ \overline{Q} & \overline{M} \end{pmatrix} \quad (8.11)$$

and

$$\begin{aligned} M_{jj} &= z + \mu_j + 2i \lambda_j t_2, \\ M_{ij} &= (\lambda_i - \lambda_j)^{-1} e^{(\lambda_i - \lambda_j) z} \quad i \neq j, \\ Q &= \text{diag} \left[\nu_i e^{-it_2 (\lambda_j^2 + \overline{\lambda_j}^2)} \right] \quad j = 1, \dots, N \end{aligned} \quad (8.12)$$

Here we are restricted to the solutions of equation (7.6). Finally one has [52]

$$\int |p|^2 dx dy = -4\pi\varepsilon N \quad . \quad (8.13)$$

The simplest solution is ($N = 1$)

$$p = 2\bar{\nu} \frac{\exp[\lambda_1 z - \bar{\lambda}_1 \bar{z} + i(\lambda_1^2 + \bar{\lambda}_1^2) t_2]}{|z + 2i\lambda_1 t_2 + \mu|^2 - \varepsilon|v|^2} \quad ,$$

$$\bar{\psi} = e^{\lambda_1 z} - (\lambda - \lambda_1)^{-1} e^{\lambda z} [\log(|z + 2i\lambda_1 t_2 + \mu|^2 - \varepsilon|v|^2)]_z \quad , \quad (8.14)$$

$$\bar{\varphi} = \frac{\varepsilon}{2} (\lambda - \lambda_1)^{-1} e^{\lambda z} p \quad .$$

These solutions give rise to surfaces and their deformations of soliton type. For them the value of the Willmore functional is

$$W = 16\pi N \quad . \quad (8.15)$$

Note that it does not depend on the sign of ε .

For the DSI equation there is another interesting class of explicit solutions. For these solutions, called dromions, the functions p and q decrease exponentially fast in all directions on the plane (ξ, η) (see *e.g.* [53,49,51]). For general dromion solution of equation (A.13) one has [54]

$$|p|^2 = -4 [\log \det (1 - \varepsilon A)]_{\xi\eta} \quad (8.16)$$

where the rectangular matrix A is of the form

$$A = \varrho \beta \varrho^\dagger \bar{\alpha} \quad (8.17)$$

and

$$(\rho)_{ij} = \rho_{ij} \quad , \quad (\rho^\dagger)_{ij} = \bar{\rho}_{ji} \quad ,$$

$$\beta_{ij} = \int_{-\infty}^{\eta} Y_i(\eta', t_2) Y_j(\eta', t_2) d\eta' \quad , \quad (8.18)$$

$$\alpha_{ij} = \int_{-\infty}^{\xi} X_i(\xi', t_2) X_j(\xi', t_2) d\xi' \quad .$$

Here X_i and Y_i are arbitrary solutions of equations

$$iX_{it_2} + X_{i\xi\xi} + u_2(\xi, t_2)X_i = 0 \quad , \quad (8.19)$$

$$iY_{it_2} + Y_{i\eta\eta} + u_1(\eta, t_2)Y_i = 0$$

and u_1 and u_2 are arbitrary functions. Choosing u_1 and u_2 as the reflectionless potentials in the Schroedinger equations (8.19), one gets multidromion solutions. For these solutions of the DSI equation one has [54]

$$\int |p|^2 d\xi d\eta = -\varepsilon \log \det (1 - \varepsilon \varrho \varrho^\dagger) \quad . \quad (8.20)$$

As a result the corresponding time-like surfaces in the Minkowski space $\mathbb{R}^{3,1}$ (the formulae (5.1),(5.2)) have the following value of the Willmore functional ($\varepsilon = 1$)

$$W = -2 \log \det (1 - \varrho \varrho^\dagger) \quad . \quad (8.21)$$

This class of surfaces could be of interest for the theory of classical strings in the Minkowski space.

The formulae (8.15) and (8.21) demonstrate us that the method of the inverse scattering transform provides us the technique to calculate the value of the Willmore functional for rather complicated surfaces in 4D spaces.

9 Particular classes of surfaces and their deformations

Here we consider some special classes of surfaces which have simple geometric characterization.

We start with minimal surfaces. In the four-dimensional spaces they are characterized by the condition $\vec{H} = 0$. Using the expressions for the mean curvature vector found in sections 3-5, we conclude that in all cases $\vec{H} = 0$ if the potentials p and q in DS-linear problems vanish. So one has

Corollary 9.1 *The formulae of section 3-5 with $p = q = 0$ give us the Weierstrass representations for minimal surfaces in 4D spaces. Space-like surfaces are parametrized by four arbitrary holomorphic functions (two holomorphic and two anti-holomorphic). Formulae for time-like surfaces contain two arbitrary functions of one variable (ξ) and two arbitrary functions of another variable (η).*

From the formulae for \vec{H}^2 follows also that surfaces with zero length of the mean curvature vector ($\vec{H}^2 = 0$) coincide with the minimal surfaces in the cases of space-like surfaces in \mathbb{R}^4 , $\mathbb{R}^{2,2}$ and time-like surfaces in $\mathbb{R}^{3,1}$. In contrast, this condition is less restrictive in the cases of space-like surfaces in $\mathbb{R}^{3,1}$ and time-like surfaces in $\mathbb{R}^{2,2}$: it is sufficient that $p = 0$ (or $q = 0$).

Superminimal immersions form a subclass of the minimal ones for which in addition the condition $\vec{X}_{zz} \cdot \vec{X}_{zz} = 0$ is satisfied (see *e.g.* [29]). For surfaces in \mathbb{R}^4 and $\mathbb{R}^{2,2}$, using (3.14) and (4.1), one gets

$$\vec{X}_{zz} \cdot \vec{X}_{zz} = \varepsilon (\bar{\psi}_{1z}\varphi_1 - \bar{\psi}_1\varphi_{1z}) (\bar{\psi}_{2z}\varphi_2 - \bar{\psi}_2\varphi_{2z}) = 0 \quad (9.1)$$

where $\varepsilon = -1$ for \mathbb{R}^4 and $\varepsilon = 1$ for $\mathbb{R}^{2,2}$.

Corollary 9.2 *The immersions of surface into \mathbb{R}^4 and $\mathbb{R}^{2,2}$ given by the Weierstrass representations (3.14)- (3.15) and (4.1)-(4.2), respectively, are superminimal if $p = 0$ and $\varphi_1 = a_1\bar{\psi}_1$ (or $\varphi_2 = a_2\bar{\psi}_2$) where a_1 (or a_2) is an arbitrary constant.*

Analogous results hold for the other cases.

The Weierstrass representations for superminimal immersions could be useful for an analysis of the problems discussed in [29].

Next geometrically interesting class of surfaces correspond to the constant mean curvature (*i.e.* $\vec{H}^2 = \text{const.}$). In the case of \mathbb{R}^4 one, obviously, has

Corollary 9.3 *Surfaces of the constant length of the mean curvature vector ($\vec{H}^2 = \text{const}$) conformally immersed into \mathbb{R}^4 are generated by the formulae*

(3.14) where $\psi_\alpha, \varphi_\alpha$ ($\alpha = 1, 2$) obey to the system of equations

$$\begin{aligned}\psi_{\alpha z} &= \frac{1}{2} \exp(i\theta_\alpha) \sqrt{\vec{H}^2 (|\psi_1|^2 + |\varphi_1|^2) (|\psi_2|^2 + |\varphi_2|^2)} \varphi_\alpha, \\ \varphi_{\alpha \bar{z}} &= -\frac{1}{2} \exp(-i\theta_\alpha) \sqrt{\vec{H}^2 (|\psi_1|^2 + |\varphi_1|^2) (|\psi_2|^2 + |\varphi_2|^2)} \psi_\alpha, \quad \alpha = 1, 2\end{aligned}\tag{9.2}$$

where $\theta_1 = -\theta_2 = \theta$ and $\theta(z, \bar{z})$ is an arbitrary function.

In the space-like case in $\mathbb{R}^{2,2}$ the surfaces of the constant \vec{H}^2 are generated by the formula (4.1) where $\psi_\alpha, \varphi_\alpha$ obey (9.2) with obvious changes of signs. Similar situation take place for time-like surfaces in $\mathbb{R}^{3,1}$ while for space-like surfaces in $\mathbb{R}^{3,1}$ and time-like surfaces in $\mathbb{R}^{2,2}$ the functions $\psi_\alpha, \varphi_\alpha$ obey the system (4.8) and (5.10) with the constraints

$$qp = -\frac{1}{4} \vec{H}^2 |\psi_1 \varphi_2 - \psi_2 \varphi_1|^2\tag{9.3}$$

and

$$qp = -\frac{1}{4} \vec{H}^2 \left[(\psi_1 \varphi_2 - \psi_2 \varphi_1) (\tilde{\psi}_1 \tilde{\varphi}_2 - \tilde{\psi}_2 \tilde{\varphi}_1) \right]^{-1}\tag{9.4}$$

respectively.

Other special classes of surfaces in 4D spaces are associated with the reductions of the corresponding linear problems. The constraint $\bar{p} = p$ in the linear systems (3.15), (4.2), (5.2) gives rise to special classes of surfaces in \mathbb{R}^4 , $\mathbb{R}^{2,2}$, and $\mathbb{R}^{3,1}$. These cases for spaces \mathbb{R}^4 and $\mathbb{R}^{2,2}$ have been considered earlier in [35]. The corresponding integrable deformations are generated by the modified Veselov-Novikov (VN) hierarchy.

Analogously, the constraint $p = q$ for the Weierstrass representations (4.7)-(4.8) and (5.9)-(5.10) gives rise to special classes of space-like surfaces in $\mathbb{R}^{3,1}$ and time-like surfaces in $\mathbb{R}^{2,2}$, respectively. The integrable deformations are generated correspondingly by the real modified Nizhnik-Veselov-Novikov (NVN) hierarchy (reduction of the DSII and DSI hierarchies under the constraint $p = q$).

Another interesting class of surfaces given by the formulae (4.7)-(4.8) and (5.9)-(5.10) is associated with the reduction $p = 1$. In this case the basic linear

system is equivalent to the equation

$$\psi_{\xi\eta} = q\psi \quad (9.5)$$

($\xi = z, \eta = \bar{z}$ for $\mathbb{R}^{3,1}$ space). Since $\psi_{\xi} = \varphi$ the Weierstrass formulae are simplified. For instance, the formulae (4.7)-(4.8) become

$$\begin{aligned} X^1 + iX^2 &= \int_{\Gamma} (\bar{\psi}_2 \psi_{1z'} dz' + \psi_1 \bar{\psi}_{2\bar{z}'} d\bar{z}') \quad , \\ X^1 - iX^2 &= \int_{\Gamma} (\bar{\psi}_1 \psi_{2z'} dz' + \psi_2 \bar{\psi}_{1\bar{z}'} d\bar{z}') \quad , \\ X^3 + X^4 &= \int_{\Gamma} (\bar{\psi}_1 \psi_{1z'} dz' + \psi_1 \bar{\psi}_{1\bar{z}'} d\bar{z}') \quad , \\ X^3 - X^4 &= - \int_{\Gamma} (\bar{\psi}_2 \psi_{2z'} dz' + \psi_2 \bar{\psi}_{2\bar{z}'} d\bar{z}') \end{aligned} \quad (9.6)$$

where

$$\psi_{\alpha z \bar{z}} = q\psi_{\alpha} \quad , \quad \alpha = 1, 2 \quad (9.7)$$

q is a real-valued function and

$$ds^2 = |w|^2 dz d\bar{z} \quad (9.8)$$

where w is the Wronskian

$$w = \begin{vmatrix} \psi_1 & \psi_{1z} \\ \psi_2 & \psi_{2z} \end{vmatrix} \quad . \quad (9.9)$$

Integrable deformations of this class of surfaces in $\mathbb{R}^{3,1}$ are generated by the VN equation (A.16) and the whole VN hierarchy.

For time-like surfaces in $\mathbb{R}^{2,2}$ the corresponding Weierstrass representation is given by

$$\begin{aligned} X^1 &= \frac{1}{2} \int_{\Gamma} [(\psi_{1\xi'} \psi_4 + \psi_{2\xi'} \psi_3) d\xi' + (\psi_1 \psi_{4\eta'} + \psi_2 \psi_{3\eta'}) d\eta'] \quad , \\ X^2 &= \frac{1}{2} \int_{\Gamma} [(\psi_{1\xi'} \psi_3 - \psi_{2\xi'} \psi_4) d\xi' + (\psi_1 \psi_{3\eta'} - \psi_2 \psi_{4\eta'}) d\eta'] \quad , \\ X^3 &= \frac{1}{2} \int_{\Gamma} [(\psi_{2\xi'} \psi_3 - \psi_{1\xi'} \psi_4) d\xi' + (\psi_2 \psi_{3\eta'} - \psi_1 \psi_{4\eta'}) d\eta'] \quad , \\ X^4 &= \frac{1}{2} \int_{\Gamma} [(\psi_{1\xi'} \psi_3 + \psi_{2\xi'} \psi_4) d\xi' + (\psi_1 \psi_{3\eta'} + \psi_2 \psi_{4\eta'}) d\eta'] \end{aligned} \quad (9.10)$$

where the real-valued functions $\psi_1, \psi_2, \psi_3, \psi_4$ satisfy the equation

$$\psi_{\alpha\eta\xi} = q\psi_{\alpha} \quad , \quad \alpha = 1, 2, 3, 4 \quad (9.11)$$

with real-valued q . Here we denote $\tilde{\varphi}_1, \tilde{\varphi}_2$ from (5.10) as $\tilde{\varphi}_1 = \psi_3, \tilde{\varphi}_2 = \psi_4$. The induced metric is

$$ds^2 = (\psi_{1\xi}\psi_2 - \psi_1\psi_{2\xi})(\psi_{3\eta}\psi_4 - \psi_3\psi_{4\eta})d\xi d\eta \quad . \quad (9.12)$$

Integrable deformations are generated by the Nizhnik equation (equation (A.16) with real-valued ξ and η) and by the whole Nizhnik hierarchy.

The Weierstrass formulae (9.6) and (9.10) represent surfaces in the spaces $\mathbb{R}^{3,1}$ and $\mathbb{R}^{2,2}$ via four solutions of the two-dimensional Schroedinger (Moutard, perturbed string) equation (9.11) (or (9.7)). So, they are the sort of four-dimensional extensions of the Lelievre formula (2.11)-(2.12). But now surfaces generated are parametrized by the minimal lines instead of asymptotic lines in (2.11)-(2.12). The fact that integrable deformations of surfaces in 4D spaces represented by (9.6) and (9.10) and of affine surfaces generated by the Lelievre formula (2.11)-(2.12) are governed by the same NVN hierarchy suggests a certain connection between these classes of surfaces.

10 One dimensional reductions

All the Weierstrass formulae presented above have one natural special case when the potentials (coefficients) in the basic linear systems depend only on one variable, say x ($x = \operatorname{Re} z$ or $x = \frac{\xi+\eta}{2}$). Let us consider space-like surfaces conformally immersed in \mathbb{R}^4 , $\mathbb{R}^{2,2}$ or $\mathbb{R}^{3,1}$. Let $p = p(x)$, $q = q(x)$. In this case solutions ψ, φ of the basic linear system are of the form

$$\psi = \chi(x) \exp(\lambda y) \quad , \quad \varphi = \tilde{\chi}(x) \exp(\lambda y) \quad (10.1)$$

where λ is an arbitrary complex parameter and $y = \operatorname{Im}(z)$ or $y = (\xi - \eta)/2$. For the second linear problem the functions ψ_2, φ_2 also have the form (10.1) but, in general, with another parameter μ . Correspondingly, the basic linear system

is reduced to the following one-dimensional one

$$\begin{pmatrix} \chi \\ \tilde{\chi} \end{pmatrix}_x = i\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \chi \\ \tilde{\chi} \end{pmatrix} + 2 \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \tilde{\chi} \end{pmatrix} . \quad (10.2)$$

The AKNS hierarchy of 1+1-dimensional integrable equations (A.18) is associated with the spectral problem (10.2). The properties of this system and, consequently, of the one-dimensional reduction of the Weierstrass representation are different for the cases of real λ and imaginary λ .

Let us consider first the case of real λ and μ . For the real λ and μ the formulae (3.14), (4.1) and (4.7) imply that

$$\vec{X}(x, y) = \vec{Y}(x) \exp[(\lambda + \mu)y] . \quad (10.3)$$

Equivalently

$$(\partial_{\bar{z}} - \partial_z) \vec{X} = i(\lambda + \mu) \vec{X} . \quad (10.4)$$

Hence

$$(\partial_{\bar{z}} - \partial_z) \vec{X} \cdot (\partial_{\bar{z}} - \partial_z) \vec{X} = -(\lambda + \mu)^2 \vec{X} \cdot \vec{X} . \quad (10.5)$$

Since for conformal immersion

$$\partial_z \vec{X} \cdot \partial_z \vec{X} = \partial_{\bar{z}} \vec{X} \cdot \partial_{\bar{z}} \vec{X} = 0 , \quad (10.6)$$

equation (10.4) implies

$$\partial_z \vec{X} \cdot \partial_{\bar{z}} \vec{X} = \frac{1}{2}(\lambda + \mu)^2 \vec{X} \cdot \vec{X} . \quad (10.7)$$

Due to (3.4), (3.6), (4.3) and (4.9)

$$\partial_z \vec{X} \cdot \partial_{\bar{z}} \vec{X} = \frac{1}{2} \det \Phi_1 \det \Phi_2 \quad (10.8)$$

where $\det \Phi_1$ and $\det \Phi_2$ are the determinants of the 2×2 matrix solutions of the form (4.13), (4.14), (4.15) for the systems (3.15), (4.1), (4.8). Using (10.1) and (10.2), one gets from (10.7) and (10.8) the relation

$$\det \hat{\Phi}_1 \det \hat{\Phi}_2 = (\lambda + \mu)^2 \vec{Y}(x) \cdot \vec{Y}(x) \quad (10.9)$$

where $\widehat{\Phi}_1$ and $\widehat{\Phi}_2$ are 2×2 matrix solutions of the system (10.2) and of the corresponding second system, respectively. It is easy to check that for the problem (10.2) $(\det \widehat{\Phi})_x = 0$. Without loss of generality one can put $\det \widehat{\Phi}_1 = \det \widehat{\Phi}_2 = 1$ (that is typical for the one-dimensional spectral problems). Consequently, the relation (10.9) gives

$$\vec{Y}(x) \cdot \vec{Y}(x) = \frac{1}{(\lambda + \mu)^2} \quad . \quad (10.10)$$

Surface with coordinates \vec{X} of the form (10.2) with real $\lambda + \mu$ is of a cone type. It is obtained by homothety of the curve with coordinate $\vec{Y}(x)$ by the factor $\exp[(\lambda + \mu)y]$. The relation (10.10) shows that this curve lies on the sphere \mathbb{S}^3 of the radius $\frac{1}{(\lambda + \mu)}$ at the original space \mathbb{R}^4 or on the hyperboloids for spaces $\mathbb{R}^{2,2}$ and $\mathbb{R}^{3,1}$.

Integrable deformations of surfaces and, consequently, of the curves with the coordinates $\vec{Y}(x)$ are given by the AKNS hierarchy. In particular, in the case of the space \mathbb{R}^4 ($q = \bar{p}$) one has an integrable motion of curves on the sphere \mathbb{S}^3 of the radius $\frac{1}{(\lambda + \mu)}$ which is governed by the NLS hierarchy. So we reproduce the result of the paper [55] about integrable motion of curves on \mathbb{S}^3 . The case $\lambda + \mu = 0$ corresponds to the integrable motion of space curves in \mathbb{R}^3 [56].

For pseudo-Euclidean spaces $\mathbb{R}^{2,2}$ and $\mathbb{R}^{3,1}$ one has integrable motions of curves on the three-dimensional hyperboloids governed by the NLS hierarchy ($\varepsilon = 1$) and by the real AKNS hierarchy ($\mathbb{R}^{3,1}$).

Thus the one-dimensional limit of the Weierstrass formulae and corresponding deformations reproduces the results for integrable motion of curves in three-dimensional spaces.

A different situation arises in the case of pure imaginary $\lambda = i\omega$ in (10.1). Indeed, for instance, for surfaces immersed into \mathbb{R}^4 with such ψ_α and φ_α , one has (see eq. (3.3))

$$\begin{aligned} \partial(X^1 + iX^2) &= -\tilde{\chi}_1 \tilde{\chi}_2 e^{i\omega + y} \quad , \quad \partial(X^1 - iX^2) = \bar{\chi}_1 \bar{\chi}_2 e^{-i\omega + y} \quad , \\ \partial(X^3 + iX^4) &= \tilde{\chi}_1 \bar{\chi}_2 e^{i\omega - y} \quad , \quad \partial(X^3 - iX^4) = \bar{\chi}_1 \tilde{\chi}_2 e^{-i\omega - y} \end{aligned} \quad (10.11)$$

where $\omega_{\pm} = \omega_1 \pm \omega_2$ and $\chi_i, \tilde{\chi}_i$ depend only on x . The induced metric is

$$ds^2 = A(x) dz d\bar{z} = (|\chi_1|^2 + |\tilde{\chi}_1|^2) (|\chi_2|^2 + |\tilde{\chi}_2|^2) dz d\bar{z} \quad . \quad (10.12)$$

So this reduction gives rise to a surface of "revolution" in \mathbb{R}^4 . The linear problem (10.2) in this case and the corresponding NLS hierarchy are studied in great details (see *e.g.* [49,57,58]). We will apply all these results to surfaces of revolution in \mathbb{R}^4 in a separate paper.

Appendix. DS hierarchy.

Here we present some basic known facts about the DS equation and the DS hierarchy. They are associated (see *e.g.* [49-51]) with the following two-dimensional linear system (Dirac equation)

$$\begin{aligned} \psi_{\xi} &= p\varphi \\ \varphi_{\eta} &= q\psi \end{aligned} \quad (A.1)$$

where ψ, φ, p, q are, in general, complex-valued functions of the independent variables ξ, η which can be either real or complex. In soliton theory this system is known as the Davey-Stewartson (DS) linear problem. An infinite hierarchy of nonlinear differential equations associated with (A.1) is referred as the DS hierarchy. It arises as the compatibility condition of (A.1) with the systems [49-51]

$$\begin{aligned} \psi_{t_n} &= A_n \psi + B_n \varphi \quad , \\ \varphi_{t_n} &= C_n \psi + D_n \varphi \end{aligned} \quad (A.2)$$

where times t_n are new (deformation) variables and A_n, B_n, C_n, D_n are differential operators of n -th order. At $n = 1$ one gets the linear system

$$\begin{aligned} p_{t_1} &= \alpha p_{\eta} + \gamma p_{\xi} \quad , \\ q_{t_1} &= \gamma q_{\xi} + \alpha q_{\eta} \end{aligned} \quad (A.3)$$

where α, γ are arbitrary constants. The corresponding operators in (A.3) are

$$A_1 = \alpha \partial_{\eta} \quad , \quad B_1 = \gamma p \quad , \quad C_1 = \alpha q \quad , \quad D_1 = \gamma \partial_{\xi} \quad (A.4)$$

Higher equations are nonlinear ones. At $n = 2$ one has the system [49]-[51]

$$\begin{aligned} p_{t_2} &= \alpha_2 (p_{\xi\xi} + p_{\eta\eta} + up) ; , \\ q_{t_2} &= -\alpha_2 (q_{\xi\xi} + q_{\eta\eta} + uq) ; , \\ u_{z\bar{z}} &= -2(pq)_{\xi\xi} - 2(pq)_{\eta\eta} \end{aligned} \quad (\text{A.5})$$

where α_2 is an arbitrary constant. For the system (A.5)

$$\begin{aligned} A_2 &= \alpha_2 (\partial_\eta^2 + w_1) , \quad B_2 = \alpha_2 (p_\xi - p\partial_\xi) , \\ C_2 &= -\alpha_2 (q_\eta - q\partial_\eta) , \quad D_2 = -\alpha_2 (\partial_\xi^2 + w_2) \end{aligned} \quad (\text{A.6})$$

where

$$w_{1\xi} = -2(pq)_\eta , \quad w_{2\eta} = -2(pq)_\xi , \quad u = w_1 + w_2 .$$

For the t_3 deformations one has (see *e.g.* [50])

$$\begin{aligned} p_{t_3} &= p_{\xi\xi\xi} + p_{\eta\eta\eta} + 3p_\xi \partial_\eta^{-1}(pq)_\xi + 3p_\eta \partial_\xi^{-1}(pq)_\eta + 3p \partial_\eta^{-1}(qp_\xi)_\xi + 3q \partial_\xi^{-1}(qp_\eta)_\eta , \\ q_{t_3} &= q_{\xi\xi\xi} + q_{\eta\eta\eta} + 3q_\xi \partial_\eta^{-1}(pq)_\xi + 3q_\eta \partial_\xi^{-1}(pq)_\eta + 3q \partial_\eta^{-1}(pq_\xi)_\xi + 3q \partial_\xi^{-1}(pq_\eta)_\eta . \end{aligned} \quad (\text{A.7})$$

In this case

$$\begin{aligned} A_3 &= \partial_\eta^3 + 3 \left[\partial_\xi^{-1}(pq)_\eta \right] \partial_\eta + 3 \left[\partial_\xi^{-1}(qp_\eta)_\eta \right] , \\ B_3 &= -p\partial_\xi^2 + p_\xi \partial_\xi - p_{\xi\xi} - 3p \left[\partial_\eta^{-1}(pq)_\xi \right] , \\ C_3 &= -q\partial_\eta^2 + q_\eta \partial_\eta - q_{\eta\eta} - 3q \left[\partial_\xi^{-1}(pq)_\eta \right] , \\ D_3 &= \partial_\xi^3 + 3 \left[\partial_\eta^{-1}(pq)_\xi \right] \partial_\xi + 3 \left[\partial_\eta^{-1}(pq_\xi)_\xi \right] . \end{aligned} \quad (\text{A.8})$$

For t_n -evolution the operators A_n, D_n are of the order n and B_n, C_n are of the order $n - 1$.

The system (A.5) is referred as the DS system. The DS hierarchy for real ξ and η is referred as the DSI hierarchy while at the case $\xi = z, \eta = \bar{z}$ it is called DSII hierarchy.

The DS hierarchy, known also as the two-component KP hierarchy, can be written in different compact forms. First, within the Sato approach it is equivalent to the infinite system of equations [59]

$$\frac{\partial Q}{\partial t_{n\alpha}} = Q (Q^{-1} H_\alpha \partial^n Q)_- , \quad \alpha = 1, 2 , \quad n = 1, 2, \dots \quad (\text{A.9})$$

where the pseudo-differential operator Q is $Q = 1 + w_1 \partial^{-1} + w_2 \partial^{-2} + \dots$, w_k are 2×2 matrices, $\partial = \partial_{t_1}$, the matrices H_α form a basis of the diagonal 2×2 matrices and $\mathcal{L}_- = \mathcal{L} - \mathcal{L}_+$ where \mathcal{L}_+ is the differential part of the operator \mathcal{L} .

Second, the DS hierarchy can be written with the use of the bilocal recursion operator $L(x, y, t_n)$ as [60] ($\xi = x + y$, $\eta = x - y$ or $\xi = x + iy$, $\eta = x - iy$)

$$\frac{\partial P}{\partial t_{n_\alpha}} = \Delta \beta_\alpha L^n (H_\alpha P' - P H_\alpha) \quad , \quad \alpha = 1, 2 \quad , \quad n = 1, 2, \dots \quad (\text{A.10})$$

where

$$P' = P(x, y', t_n) = \begin{pmatrix} 0 & p(x, y', t) \\ q(x, y', t) & 0 \end{pmatrix} \quad ,$$

β_α are arbitrary constants, the bilocal recursion operator L acts as follows

$$L\chi(x, y', y') = (\partial_x + \sigma_3 \partial_y) \tilde{\chi} + \partial_{y'} \tilde{\chi}' \sigma_3 - P d^{-1} (P \tilde{\chi} - \tilde{\chi} P')_D + d^{-1} (P \tilde{\chi} - \tilde{\chi} P')_D P'(x, y)$$

where $[\sigma_3, \tilde{\chi}] = \chi$, $d = \partial_x + \sigma_3 (\partial_y + \partial_{y'})$ and Z_D means the diagonal part of the 2×2 matrix Z . The bilocality of the recursion operator is the principal feature of the 2+1 integrable systems.

The equations from the DS hierarchy are integrable by the inverse spectral transform method (see [49-51]). They possess all remarkable properties typical for the 2+1-dimensional soliton equations, namely: there are infinite classes of solutions given by explicit formulae (solutions with functional parameters, multisoliton solutions, periodic solutions), infinite symmetry algebra, Darboux and Backlund transformations, Hamiltonian and Lagrangian structures etc. . They have an infinite set of integrals of motion C_n (*i.e.* $\frac{\partial C_n}{\partial t_m} = 0$, $n, m = 1, 2, \dots$). The simplest of them is

$$C_1 = \int q p d\xi d\eta \quad (\text{A.11})$$

while higher integrals of motion are non-local. Emphasize that (A.11) is the integral of motion for the whole DS hierarchy ($\frac{\partial C_1}{\partial t_m} = 0$, $m = 1, 2, \dots$).

The DS hierarchy contains different sub-hierarchies associated with different specializations of p and q (reductions). The most important reduction is $q = \varepsilon \bar{p}$

where $\varepsilon = \pm 1$. This reduction is compatible with all equations of the DS hierarchy if one chooses

$$\begin{aligned} A_{2n-1} &= \partial_\eta^{2n-1} + \dots, & D_{2n-1} &= \partial_\xi^{2n-1} + \dots, \\ A_{2n} &= \pm i \partial_\eta^{2n} + \dots, & D_{2n} &= \mp i \partial_\xi^{2n} + \dots \end{aligned} \quad (n = 1, 2, 3, 4). \quad (\text{A.12})$$

In particular, the t_2 -flow takes a form

$$\begin{aligned} p_{t_2} &= i(p_{\xi\xi} + p_{\eta\eta} + up), \\ u_{\xi\eta} &= -2\varepsilon|p|_{\xi\xi}^2 - 2\varepsilon|p|_{\eta\eta}^2 \end{aligned} \quad (\text{A.13})$$

for which

$$\begin{aligned} \psi_{t_2} &= i(\partial_\eta^2 + w_1)\psi + i(p_\xi - p\partial_\xi)\varphi, \\ \varphi_{t_2} &= -i\varepsilon(\bar{p}_\eta - \bar{p}\partial_\eta)\psi - i(\partial_\xi^2 + w_2)\varphi \end{aligned} \quad (\text{A.14})$$

where $w_{1\xi} = -2\varepsilon|p|_\eta^2$, $w_{2\eta} = 2\varepsilon|p|_\xi^2$.

The t_3 deformation is given now by the equation

$$p_{t_3} = p_{\xi\xi\xi} + p_{\eta\eta\eta} + 3\varepsilon p_\xi \partial_\eta^{-1}(|p|_\xi^2) + 3\varepsilon p_\eta \partial_\xi^{-1}(|p|_\eta^2) + 3\varepsilon p \partial_\eta^{-1}(\bar{p}p_\xi)_\xi + 3\varepsilon p \partial_\xi^{-1}(\bar{p}p_\eta)_\eta \quad (\text{A.15})$$

where the evolution of ψ and φ is given by (A.2), (A.8) with $q = \varepsilon\bar{p}$.

Equation (A.13) is the proper DS equation which appears in hydrodynamics and plasma physics (see [49]). At $\varepsilon = -1$ and $\varepsilon = 1$ one has the defocusing and focusing cases, respectively.

Deeper reductions of the DS hierarchy are associated with constraints $p = \pm q$ or $p = 1$. Both of them are admissible only by equations with odd times. Under the reduction $p = 1$ the system (A.7) is converted into the equation

$$q_{t_3} = q_{\xi\xi\xi} + q_{\eta\eta\eta} + 3[q\partial_\eta^{-1}(q_\xi)]_\xi + 3[q\partial_\xi^{-1}(q_\eta)]_\eta \quad (\text{A.16})$$

and the linear system (A.1) is equivalent to

$$\psi_{\xi\eta} = q\psi. \quad (\text{A.17})$$

Equation (A.16) is the well-known Nizhik-Veselov-Novikov (NVN) equation. Under this reduction the DS hierarchy is converted into the NVN hierarchy.

In the one-dimensional limit $p_\xi = p_\eta$, $q_\xi = q_\eta$, *i.e.* $p = p(x) = p\left(\frac{\xi+\eta}{2}\right)$, $q = q(x) = q\left(\frac{\xi+\eta}{2}\right)$, the DS hierarchy is reduced to the AKNS hierarchy of the 1+1-dimensional integrable system. Using the recursion operator one can represent this hierarchy in the form (see *e.g.* [49])

$$\begin{pmatrix} q \\ -p \end{pmatrix}_{t_n} = \alpha_n L^n \begin{pmatrix} q \\ p \end{pmatrix}, \quad n = 1, 2, \dots \quad (\text{A.18})$$

where α_n are constants and the recursion operator L is

$$L = \begin{pmatrix} \partial_x - 2q\partial_x^{-1}p & 2q\partial_x^{-1}q \\ -2p\partial_x^{-1}p & -\partial_x + 2p\partial_x^{-1}q \end{pmatrix}. \quad (\text{A.19})$$

The simplest equations from the AKNS hierarchy look like

$$\begin{aligned} q_{t_2} &= \alpha_2 (q_{xx} - 2q^2p), \\ p_{t_2} &= -\alpha_2 (p_{xx} - 2p^2q) \end{aligned} \quad (\text{A.20})$$

and

$$\begin{aligned} q_{t_3} &= \alpha_3 [q_{xxx} - 6pqq_x], \\ p_{t_3} &= \alpha_3 [p_{xxx} - 6qpp_x]. \end{aligned} \quad (\text{A.21})$$

Under the reduction $q = \varepsilon \bar{p}$ the system (A.20) becomes ($\alpha_2 = i$)

$$ip_{t_2} = p_{xx} - 2\varepsilon |p|^2 p \quad (\text{A.22})$$

that is the famous nonlinear Schroedinger (NLS) equation. At the case $p = -1$ ($\alpha_3 = -1$) the system (A.21) is reduced to the celebrated Korteweg de Vries (KdV) equation

$$q_{t_3} = -q_{xxx} - 6qq_x \quad (\text{A.23})$$

while under the reduction $q = p$ ($\alpha_3 = -1$) one gets the modified KdV equation

$$p_{t_3} = p_{xxx} - 6p^2 p_x. \quad (\text{A.24})$$

Equations (A.13), (A.16) and (A.15) ($\bar{p} = p$) are the 2+1-dimensional integrable generalizations of the NLS, KdV and mKdV equations, respectively.

Acknowledgements. One of the authors (B.G.K.) appreciates very much the discussions with E. Ferapontov, F. Pedit, U. Pinkall and T. Willmore. B.G.K. is also grateful to D. Fairlie for attracting his attention to the paper [46].

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